

On Minima of Discrimination Functions

Janne V. Kujala

Ehtibar N. Dzhafarov*

University of Jyväskylä

Purdue University

Abstract

A discrimination function $\psi(x, y)$ assigns a measure of discriminability to stimulus pairs x, y (e.g., the probability with which they are judged to be different in a same-different judgment scheme). If for every x there is a single y least discriminable from x , then this y is called the point of subjective equality (PSE) for x , and the dependence $h(x)$ of the PSE for x on x is called a PSE function. The PSE function $g(y)$ is defined in a symmetrically opposite way. If the graphs of the two PSE functions coincide (i.e., $g \equiv h^{-1}$), the function is said to satisfy the Regular Minimality law. The minimum level functions are restrictions of ψ to the graphs of the PSE functions. The conjunction of two characteristics of ψ , (1) whether it complies with Regular Minimality, and (2) whether the minimum level functions are constant, has consequences for possible models of perceptual discrimination. By a series of simple theorems and counterexamples, we establish set-theoretic, topological, and analytic properties of ψ which allow one to relate to each other these two characteristics of ψ .

KEYWORDS: minimum level function, perceptual discrimination, subjective equality, Regular Minimality, stimulus space, well-behaved discrimination function

1. Introduction

The principal object of consideration in this paper is *discrimination probability function*,

$$\begin{aligned}\psi(x, y) &= \Pr[x \text{ and } y \text{ are judged to be different}] \\ &= 1 - \Pr[x \text{ and } y \text{ are judged to be the same}].\end{aligned}$$

An experimental paradigm must involve a stimulus characteristic in which all x -stimuli differ from all y -stimuli, allowing the experimenter to treat (x, y) as an ordered pair (i.e., distinguish (a, b) from (b, a) and treat (a, a) as a pair rather than a single stimulus). Using the term coined in Dzhafarov (2002), x and y belong to two *observation areas*, whose difference, if perceived, is to be ignored by the observer in choosing between “same” and “different.” In a traditional psychophysical experiment, say, comparison of aperture

*Corresponding author. Department of Psychological Sciences, Purdue University, 703 Third Street, West Lafayette, IN 47907-2081, USA. E-mail address: ehtibar@purdue.edu (E.N. Dzhafarov).

colors varying in color coordinates, the observation area is usually determined by the chronological order of the two stimuli (first or second), or by their spatial locations (say, left or right with respect to a fixation point). Thus, an x -stimulus and a y -stimulus may be described as, respectively, (left, color coordinates r, g, b) and (right, color coordinates r', g', b'). Formally, $x \in X$ and $y \in Y$,

$$\psi : X \times Y \rightarrow [0, 1], \quad (1)$$

where X and Y are different stimulus spaces, even if the variable parameters of stimuli (such as color coordinates) have one and the same set of possible values. Figure 1 illustrates the logic of the situation on an example where the observation area, X or Y , is determined by the mouth line curvature (“happy” versus “sad”). For the variety of experimental paradigms and types of stimuli, as well as the variety of meanings for the categories “same” and “different,” see Dzhafarov and Colonius (2006).

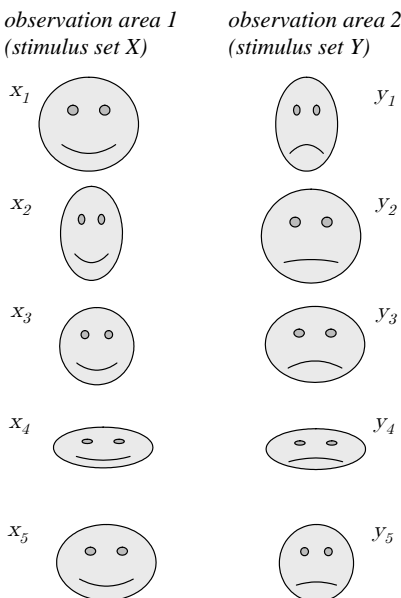


Figure 1. A sample of faces presented pairwise with the question: Is this one and the same “person”? In all pairs, one of the two faces is “happy” (and is considered a stimulus from X), the other is “sad” (a stimulus from Y). The variable parameters of the stimuli are the shapes of the ovals for face and eyes (having one and the same set of values for x and y stimuli). To illustrate the notion of a point of subjective equality (PSE): if the pair (x_4, y_4) evokes the response “same” more frequently than any pair (x_4, y) , $y \neq y_4$, then y_4 is the PSE for x_4 ; if the pair (x_4, y_4) evokes the response “same” more frequently than any pair (x, y_4) , $x \neq x_4$, then x_4 is the PSE for y_4 .

In spite of our primary interest in discrimination probabilities, throughout this paper we make no use of the fact that ψ is *probability*. This means that our discussion equally pertains to any function $\psi(x, y)$ which assigns a “degree of discriminability” to stimulus pairs (x, y) , with the values ranging from zero (x and y are indistinguishable) to a positive number, which can always be taken to be 1, indicating a “perfect discriminability.” A prototypical example is provided by numerical ratings of dissimilarity on some “from-to” scale. With this generalization in mind, we refer to any function ψ of the form (1) as *discrimination function*,

with probability being merely one of its interpretations, albeit the main one.

Stimulus spaces (observation areas) X and Y are treated in this paper on a very general level. The strongest constraint we impose on them is that they are arc-connected, first countable, Hausdorff spaces. We defer to the concluding section a brief discussion of where such topological properties are derived from.

Consider a discrimination function ψ which has the following properties:

($\mathcal{P}1$) for some function $h : X \rightarrow Y$ and all $x \in X, y \in Y$,

$$y \neq h(x) \implies \psi(x, h(x)) < \psi(x, y), \quad (2)$$

($\mathcal{P}2$) for some function $g : Y \rightarrow X$ and all $x \in X, y \in Y$,

$$x \neq g(y) \implies \psi(g(y), y) < \psi(x, y). \quad (3)$$

Then, for any $x \in X$, the stimulus $h(x) \in Y$ is called the *point of subjective equality* (PSE) for x , and for any $y \in Y$, the stimulus $g(y) \in X$ is called the PSE for y . The functions h and g are referred to as the *PSE functions* ($X \rightarrow Y$ and $Y \rightarrow X$, respectively).

The function $\omega_h : X \rightarrow [0, 1]$ defined by

$$\omega_h(x) = \psi(x, h(x)) \quad (4)$$

and the function $\omega_g : Y \rightarrow [0, 1]$ defined by

$$\omega_g(y) = \psi(g(y), y) \quad (5)$$

are called the *minimum level functions* (along the PSE functions h and g , respectively).

Given $\mathcal{P}1$ and $\mathcal{P}2$, suppose that ψ also satisfies the following condition:

($\mathcal{P}3$) for all $x \in X, y \in Y$, y is the PSE for x if and only if x is the PSE for y . That is,

$$g \equiv h^{-1}. \quad (6)$$

Then we say that ψ satisfies the law of *Regular Minimality*. In this case the PSE functions h and g define one and the same set of ordered pairs (x, y) , and it is convenient to speak, by abuse of language, of a *single* PSE function (equivalently written as either h or g) and, correspondingly, of a single minimum level function (written as ω_h or ω_g). Clearly, if ψ satisfies the law of Regular Minimality, both h and g are bijective functions, and X and Y are in a one-to-one correspondence.¹

Assuming that ψ satisfies $\mathcal{P}1$ and $\mathcal{P}2$, in this paper we focus on the following two characteristics:

¹The terminology just introduced follows Dzhafarov (2002, 2003, 2006) and Dzhafarov and Colonius (2005, 2006, 2007), with one important difference: we predicate the notions of the PSE function and the minimum level function on the properties $\mathcal{P}1$ and $\mathcal{P}2$ alone, rather than on the law of Regular Minimality in its entirety. In other words, we allow for two distinct PSE functions and two minimum level functions.

1. whether the minimum level functions, ω_h or ω_g , are constant (both or one of them);
and
2. whether ψ satisfies the law of Regular Minimality (i.e., satisfies $\mathcal{P}3$ in addition to $\mathcal{P}1$ - $\mathcal{P}2$).

These characteristics are important because the conjunction of Regular Minimality with the nonconstancy of the minimum level function (*Nonconstant Self-Dissimilarity*) has been shown to impose nontrivial constraints on the basic modeling schemes for the process of perceptual discrimination (Dzhafarov, 2003, 2006; Dzhafarov & Colonius, 2006; Ennis, 2006).

We know that $\mathcal{P}1$ and $\mathcal{P}2$ do not imply Regular Minimality, and that Regular Minimality can be satisfied with or without the minimum level function $\omega_h(x)$ (or $\omega_g(y)$), as in this case the two are interchangeable being constant. Figure 2 (Regular Minimality with constant ω_h), Fig. 3 (Regular Minimality with nonconstant ω_h), and Fig. 4 ($\mathcal{P}1$ - $\mathcal{P}2$, but not $\mathcal{P}3$, with nonconstant ω_h and ω_g) are variants of the examples presented, with various degree of detail, in Dzhafarov (2002, 2003, 2006) and Dzhafarov and Colonius (2005, 2006).²

It would be premature, however, to conjecture that the constancy of ω_h and/or ω_g on the one hand and Regular Minimality on the other have nothing to do with each other. The two are related in a variety of ways, most of which are best characterized as *simple but not immediately obvious*. The explication of these simple but nonobvious relations between the two properties is the main goal of this paper.

Consider, for instance, a function $\psi :]0, 1[\times]0, 1[\rightarrow [0, 1]$ with the PSE functions as shown in Fig. 5. It is easy to show that ω_h in such a case cannot be constant. Indeed, $\psi_1 < \psi_2$ by the definition of $h(x)$, and $\psi_2 < \psi_3$ by the definition of $g(y)$, whence $\psi_1 < \psi_3$. That ω_g cannot be constant is shown analogously.

In Section 2 we generalize and qualify this observation by establishing propositions of the form

(Const \rightarrow RegMin): constancy of minimum level function(s) ω_h and/or $\omega_g \implies$ Regular Minimality

for broad subclasses of discrimination functions ψ subject to $\mathcal{P}1$ - $\mathcal{P}2$. The subclasses of ψ are defined in set-theoretic or topological terms: the strongest constraint we consider is that X and Y are Hausdorff, first-countable, connected topological sets, and that the functions ψ , h , and g are continuous mappings (in which case we also consider the relationship between the continuity of h , g , and ψ).

No topological properties of ψ would suffice to ensure the reverse implication

²Note that the linearity of the PSE functions in these examples is only chosen for graphical convenience: the PSE functions can be made nonlinear by homeomorphic mappings of stimulus sets X and Y onto themselves. Thus, the PSE lines in Fig. 4 can be transformed into a picture like in Fig. 5 by mapping $X = \mathbb{R}$ and $Y = \mathbb{R}$ onto $]0, 1[$. This consideration alone shows that the conformity with Regular Minimality we see in Figs. 2 and 3 must not be interpreted as the lack of “constant error” (the term traditionally used to indicate $h(x) \neq x$ and/or $g(y) \neq y$). In general, if X and Y are such that the notion of a constant error is well-defined (it need not be, as elements of X and Y are allowed to be physically non-comparable, e.g., measured in different units), the lack of Regular Minimality, as in Fig. 4, necessarily means that either $h(x) \neq x$ for some $x \in X$ or $g(y) \neq y$ for some $y \in Y$; while the adherence to Regular Minimality, as in Figs. 2 and 3, is compatible with both $h(x) \equiv x$ and $h(x) \neq x$ (Dzhafarov, 2006; Dzhafarov & Colonius, 2006).

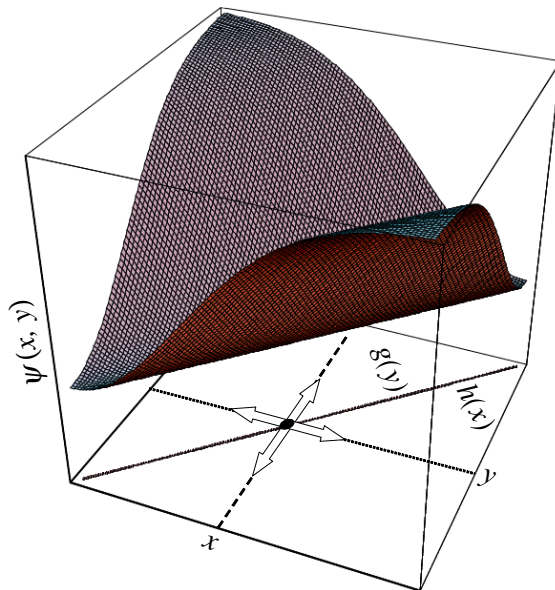


Figure 2. A discrimination probability function $\psi(x, y)$ for real-valued stimuli ($X = Y = \mathbb{R}$). (This particular example is generated by means of Luce & Galanter’s, 1963, model, with normal constant-variance distributions whose means change as smooth functions of stimuli; see Dzhafarov, 2003, for details.). The function satisfies Regular Minimality. The curve in the xy -plane is the PSE function for ψ , which can be equivalently written as $y = h(x)$ or $x = g(y)$ (in this case, linear functions): for any point taken on the PSE line, the value of ψ increases as one moves away from this point in any of the four directions shown. The minimum level function $\omega_h(x)$ (equivalently, $\omega_g(y)$) in this example is constant: the bottom line of ψ is parallel to its PSE shadow in the xy -plane.

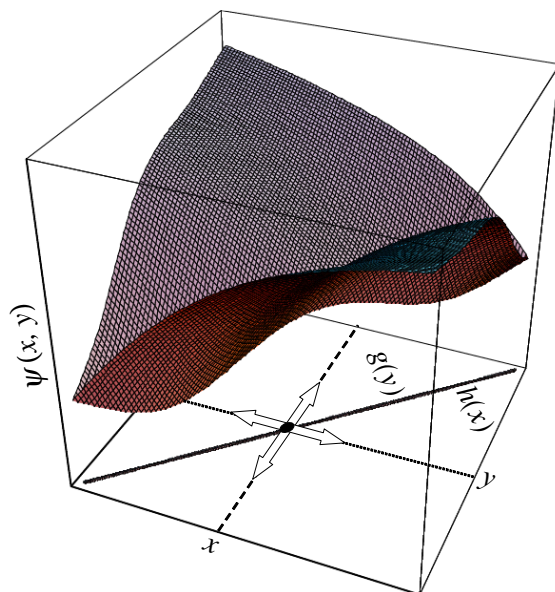


Figure 3. A discrimination probability function $\psi(x, y)$ (with X and Y intervals of \mathbb{R}) which satisfies Regular Minimality but has a nonconstant minimum level function. (This example is generated by means of the “quadrilateral dissimilarity” model; see Dzhafarov & Colonius, 2005, for details.)

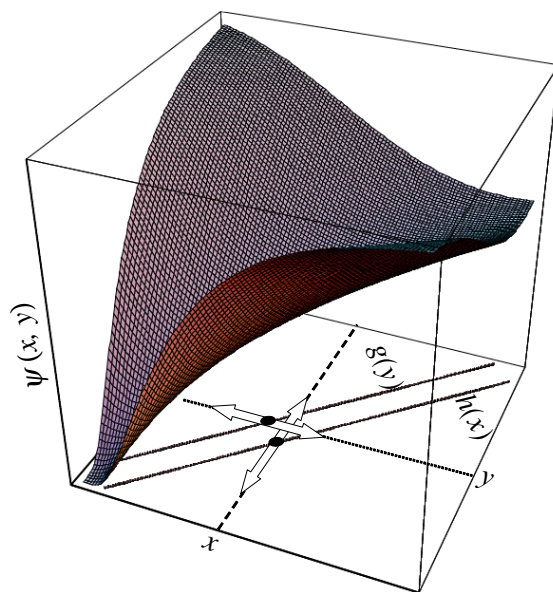


Figure 4. A discrimination probability function $\psi(x, y)$ ($X = Y = \mathbb{R}$) which does not satisfy Regular Minimality although it does conform with $\mathcal{P}1$ and $\mathcal{P}2$: the function $y = h(x)$ and $x = g(y)$ are defined for all x and y , respectively, but the two graphs (in this case, straight lines) do not coincide. As one moves away from a point on the line $y = h(x)$ in either of the two y -directions (or from a point on $x = g(y)$ in either of the two x -directions), the value of ψ increases. Although the corresponding minimum level functions are not visually discernible, it is apparent that both minimum level functions are nonconstant. (This example, like the one in Fig. 2, is generated by means of Luce & Galanter's, 1963, model, but this time the variances change together with the means as smooth functions of stimuli; see Dzhafarov, 2003, 2006, for details.)

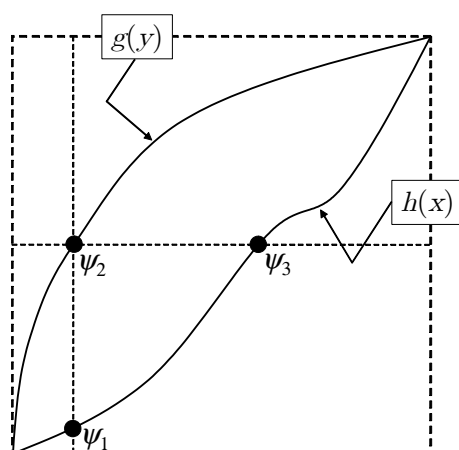


Figure 5. A pair of distinct PSE functions h and g for a discrimination function ψ . Values ψ_1, ψ_2, ψ_3 of ψ are shown for three points indicated by circles. The minimum level function ω_h cannot be constant because $\psi_1 < \psi_2 < \psi_3$.

(RegMin→Const): Regular Minimality \implies constancy of minimum level function.

Thus, in Section 3 we show that the class of continuous functions ψ subject to Regular Minimality with continuous PSE functions, if it is nonempty, includes functions with *all* possible combinations of a continuous PSE function and a continuous minimum level function. It is known, however, that Regular Minimality does imply the constancy of the minimum level function if $X, Y \subset \mathbb{R}$ and ψ and h are continuously differentiable functions:

$$\frac{d\psi(x, h(x))}{dx} = \left. \frac{\partial\psi(z, h(x))}{\partial z} \right|_{z=x} + \left. \frac{\partial\psi(x, z)}{\partial z} \right|_{z=h(x)} \frac{dh(x)}{dx} = 0, \quad (7)$$

because the two partial derivatives vanish at the minima of, respectively, $z \mapsto \psi(z, h(x))$ and $z \mapsto \psi(x, z)$. In Dzhafarov (2003) this observation is generalized to “near-smooth patches” of functions ψ defined on $X, Y \subset \mathbb{R}^n$. In Section 3 we generalize this observation further. We show how a function defined on arc-connected X, Y can be represented by a set of functions mapping $[0, 1] \times [0, 1]$ into $[0, 1]$, called *arc-parametrized facets* of ψ . The implication RegMin→Const is then related to analytic properties of these functions.

2. From Constancy to Regular Minimality

In this section we discuss properties of discrimination functions ψ which enable implications of the form Const→RegMin.

2.1. Set-theoretic properties

Our first proposition is that the constancy of both minimum level functions, ω_h and ω_g , always implies Regular Minimality.

THEOREM 1 Let ψ satisfy $\mathcal{P}1$ – $\mathcal{P}2$, and let $\omega_h(x) = c_1$ and $\omega_g(y) = c_2$ for all $x \in X$ and $y \in Y$. Then $c_1 = c_2$ and $g \equiv h^{-1}$ (i.e., ψ satisfies $\mathcal{P}3$).

Proof. Since

$$\psi(x, y) \geq \max\{\psi(x, h(x)), \psi(g(y), y)\} = \max\{c_1, c_2\}$$

for all x and y , $\max\{c_1, c_2\}$ is the global minimum of ψ . Hence

$$c_1 = c_2 = \min_{x, y} \psi(x, y).$$

Then $\psi(x, h(x)) \leq \psi(x', h(x))$ for any x, x' , and it follows from $\mathcal{P}2$ that, for any x ,

$$x = g(h(x)).$$

Analogously, $\psi(g(y), y) \leq \psi(g(y), y')$ for any y, y' , and $\mathcal{P}1$ implies that, for any y ,

$$y = h(g(x)).$$

Hence $g \equiv h^{-1}$. ■

If only one of the minimum level functions, ω_h or ω_g , is assumed to be constant, the main result is based on the following lemma, which essentially codifies the observation made in relation to Fig. 5.

LEMMA 1 Let ψ satisfy $\mathcal{P}1$ - $\mathcal{P}2$, and let, for some $x \in X$, $y = h(x)$. Then

$$\psi(x, y) = \psi(g(y), h(g(y))) \quad \text{if and only if} \quad x = g(y).$$

Proof. By the definition of h ,

$$\psi(g(y), h(g(y))) \leq \psi(g(y), y),$$

and by the definition of g ,

$$\psi(g(y), y) \leq \psi(x, y).$$

Hence

$$\psi(x, y) = \psi(g(y), h(g(y))) \implies \psi(g(y), y) = \psi(x, y).$$

Invoking the definition of g again,

$$\psi(g(y), y) = \psi(x, y) \implies x = g(y).$$

This proves the necessity (only if) statement. The sufficiency is obvious. ■

That the constancy of only one of the minimum level functions does not generally imply Regular Minimality can be seen from the example in Fig. 6. Note: although in this subsection we only deal with X, Y as unstructured sets, in this particular example we need the continuity of ψ to interpolate the values of ψ between those marked. We use intervals of reals as X and Y in all our subsequent examples too, and the continuity (of $\psi, h, g, \omega_h, \omega_g$) is understood in relation to the usual, Euclidean topology.

It is easy to surmise that the reason Regular Minimality is violated in this example is that $g(y)$ maps Y on a proper subset of X rather than on the entire X . This can be “corrected” by splitting ψ into two parts with “properly chosen” domains, as shown in Fig. 7. The next lemma and a theorem that follows from it clarify the issue. We use the notation $f|B$ for the restriction of function f defined on A to $B \subset A$.

LEMMA 2 Let ψ satisfy $\mathcal{P}1$ - $\mathcal{P}2$ with the PSE functions h and g , and let $\omega_h(x) = c$ on some $X' \subset X$ (or $\omega_g(y) = c$ on some $Y' \subset Y$). Then the restriction $\psi|X' \times h(X')$ of ψ (respectively, $\psi|g(Y') \times Y'$) satisfies $\mathcal{P}3$, with the PSE functions $h|X'$ and $g|h(X')$ (respectively, $h|g(Y')$ and $g|Y'$).

Proof. That the restriction of ψ to $X' \times h(X')$ satisfies $\mathcal{P}1$ - $\mathcal{P}2$, with the PSE functions $h|X'$ and $g|h(X')$, is obvious. Since, for all $x \in X'$,

$$\omega_h(x) = \psi(x, h(x)) = \psi(g(h(x)), h(g(h(x)))) = c,$$

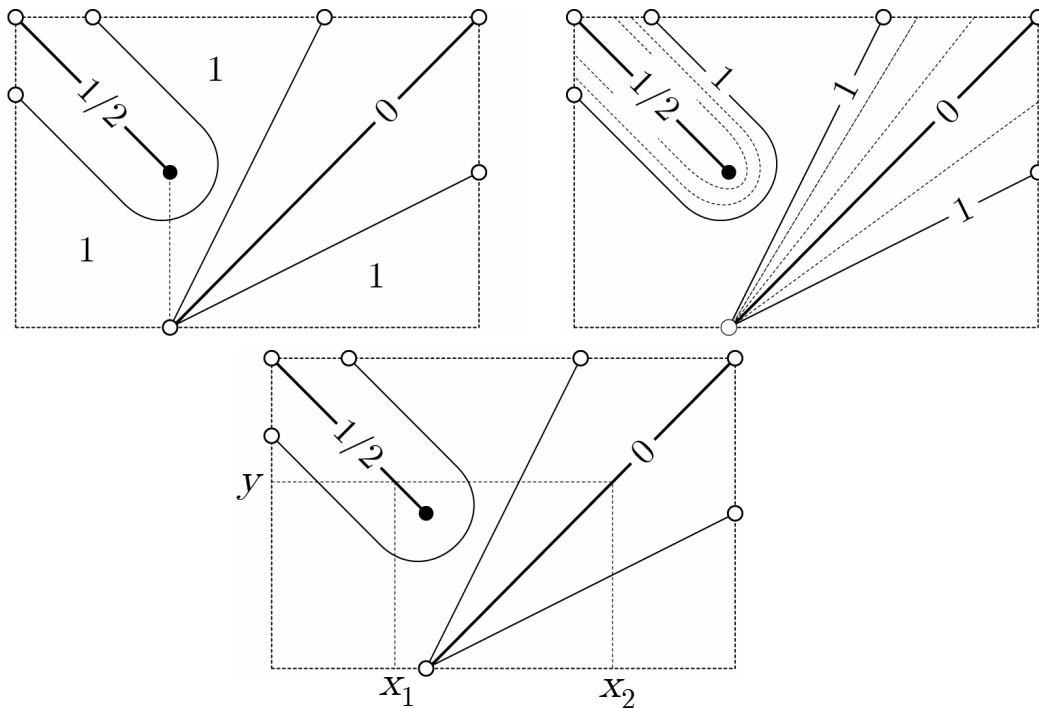


Figure 6. A contour map for a continuous discrimination function defined on a Cartesian product of two open real intervals. Closed circles indicate points that belong to the domain of ψ ; open circles fall outside the domain. The numbers indicate the elevations (values) of ψ on different areas and along lines. The intermediate values of ψ between the bottoms (heavy solid lines) and boundaries (light solid lines) of the two “canyons” can be obtained by arbitrary continuous interpolation between the solid lines, with intermediate elevation contours shown by the dashed lines in the right panel. The right-hand heavy solid line graphs the PSE function $g(y)$, with the ω_g being constant (zero). The two heavy solid lines together form a graph of the (discontinuous) PSE function $h(x)$, with a nonconstant ω_h (its value changes from $1/2$ to 0). Regular Minimality is violated because, as shown in the bottom panel, $y = h(x_1)$ but $g(y) = x_2 \neq x_1$.

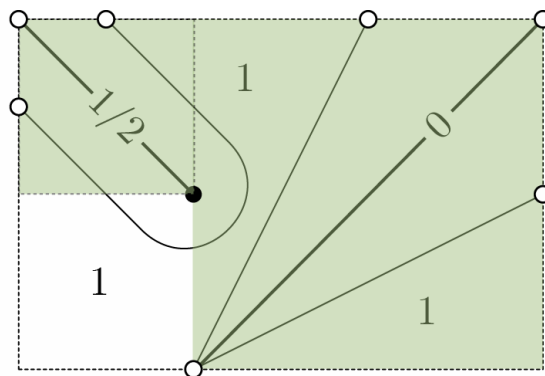


Figure 7. Two restrictions of ψ satisfying Regular Minimality. The restricted domains are shown by two shaded rectangular areas containing the zero-level line and the $1/2$ -level line as their diagonals.

where $h(x), h(g(h(x))) \in h(X')$, we invoke Lemma 1 to conclude that, for every $x \in X'$,

$$x = g(h(x)).$$

But this is the statement of $\mathcal{P}3$ for the restriction of ψ to $X' \times h(X')$. ■

THEOREM 2 Let ψ satisfy $\mathcal{P}1$ - $\mathcal{P}2$, and let $\omega_h(x) \equiv c$ (or $\omega_g(y) \equiv c$). Then ψ satisfies $\mathcal{P}3$ if and only if h maps X onto Y (respectively, g maps Y onto X).

Proof. The sufficiency (if) is a corollary to Lemma 2. The necessity (only if) is obvious: Regular Minimality implies that h is bijective. ■

A non-surjective h in the formulations of the properties $\mathcal{P}1$ and $\mathcal{P}2$ means that while every $y \in Y$ has its PSE in X , there are some $y \in Y$ which are PSEs for no $x \in X$. At least in some contexts this possibility may be deemed implausible: in view of the domain redefinition offered by Lemma 2 and illustrated in Fig. 7, one might think of a non-surjective h as a consequence of “improperly” defined stimulus sets (observation areas). It seems reasonable therefore (although we do not adopt this point of view in this paper) to consider a reformulation of $\mathcal{P}1$ and $\mathcal{P}2$ in which h and g will be required to be surjective mappings.

2.2. Topological properties

We look now into topological properties of the discrimination function ψ which enable the implication $\text{Const} \rightarrow \text{RegMin}$. Specifically, we assume that

- (C) X and Y are Hausdorff, first countable, connected topological spaces,³ and ψ is continuous with respect to the product topology on $X \times Y$.

We will refer to a function ψ with property C as a \mathcal{C} -function.

A prototypical example of a \mathcal{C} -function will be a continuous function defined on the Cartesian product of connected subsets of \mathbb{R}^n endowed with the usual topology. This was the level of consideration adopted in Dzhafarov (2002, 2003), and all our examples, being confined to intervals of \mathbb{R} , fall within this category too.

We begin by discussing the continuity of the PSE functions for a \mathcal{C} -function ψ subject to $\mathcal{P}1$ and $\mathcal{P}2$. That the continuity of h and g does not follow from the property C and has to be stipulated separately⁴ is apparent from Fig. 6, where $h(x)$ is discontinuous. The question of whether this discontinuity may be responsible for the violation of Regular Minimality in this example is answered in the negative: the example

³As an informal reminder: the first countability is needed to speak of convergence and continuity in terms of sequences of points, the Hausdorff property ensures that a sequence cannot converge to more than one limit, and the connectedness of a set means that its proper subsets cannot be both closed and open.

⁴In Dzhafarov (2003) the continuity of h and g (in \mathbb{R}^n) was part of the formulation of the properties $\mathcal{P}1$ and $\mathcal{P}2$. Although we obviously do not adopt this approach here, in view of our results the generality loss it entails (especially, in \mathbb{R}^n) is primarily of a “monster-barring” variety.

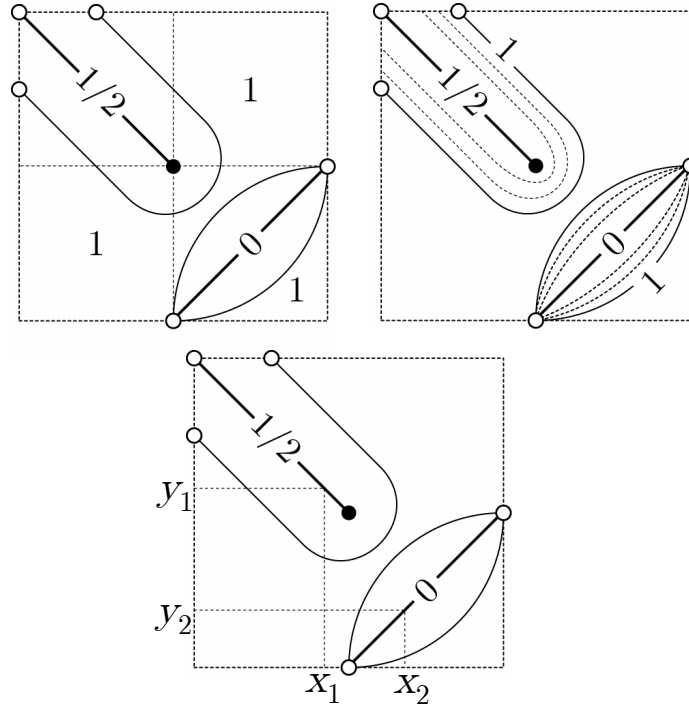


Figure 8. A contour map for a continuous discrimination function which satisfies Regular Minimality with a nonconstant minimum level function. The details are analogous to those in Fig. 6. The PSE functions h and g are discontinuous.

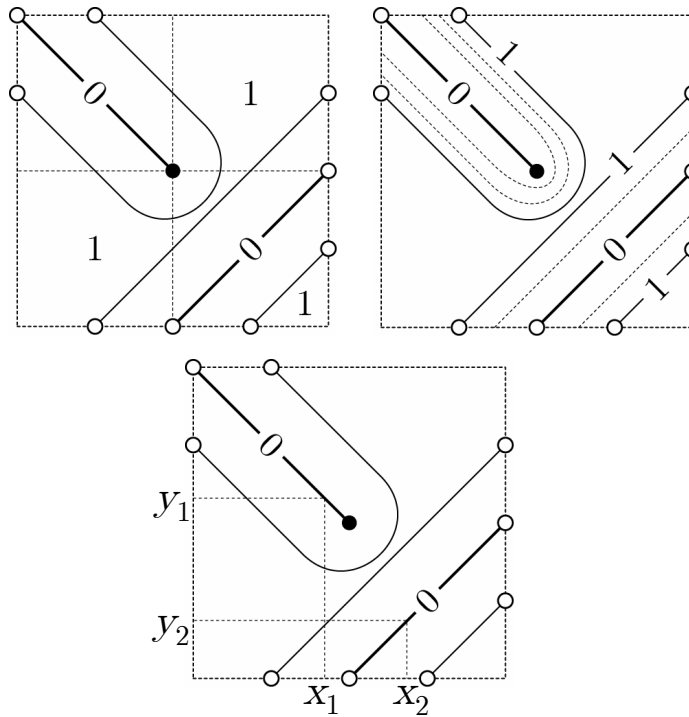


Figure 9. A contour map for a continuous discrimination function which satisfies Regular Minimality with a *constant* minimum level function. The details are analogous to those in Fig. 6. The PSE functions h and g are discontinuous.

given in Fig. 8 shows that both $h(x)$ and $g(y)$ can be discontinuous for a continuous ψ which does satisfy Regular Minimality. Figure 9 shows that this is possible even if the minimum level function (written as ω_h or ω_g) is constant.

Inspection of the examples in Figs. 6, 8, and 9 also shows, however, that the discontinuity of the PSE functions in them is of a special nature. Focusing on the function $h(x)$, let us say that a sequence $x_n \rightarrow x$ reveals a discontinuity of h at x if $h(x_n)$ does not converge to $h(x)$. In Figs. 6, 8, and 9 (left upper panels), the sequences revealing a discontinuity of $h(x)$ are all those converging from the right to the lower endpoint of the zero-level line (let us call this point a). The special nature of the revealed discontinuities is in that for any sequence $x_n \rightarrow a+$ the corresponding sequence $h(x_n)$ has no limit points in Y . It would be futile to try, as we will see, to modify the examples to force $h(x_n)$ to converge to some limit in Y other than $h(a)$. In particular, Fig. 10 demonstrates that the examples will not work if we replace the interval Y with its closure \bar{Y} preventing thereby limit points in Y from “escaping” (in the figure, the interval X is also closed, for symmetry, but this is immaterial).

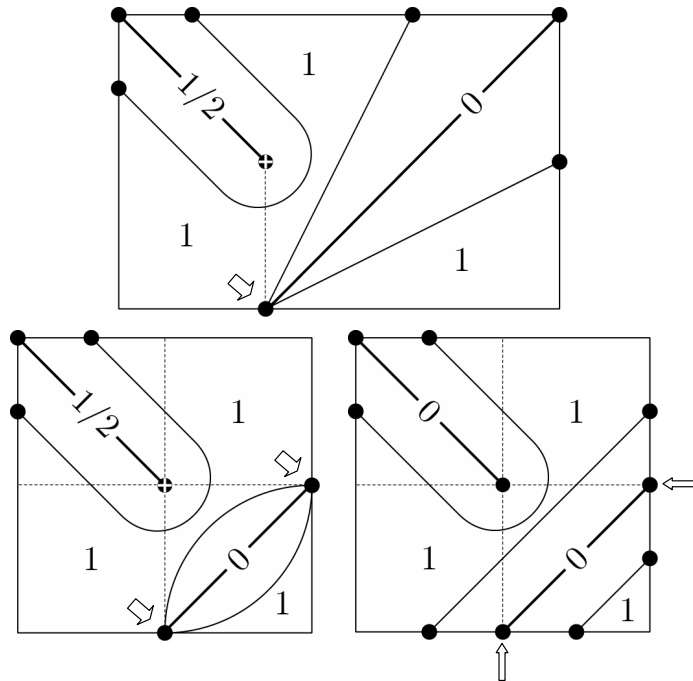


Figure 10. An attempt to modify the examples in Figs. 6, 8, and 9 (respectively, upper, left, and right panels) by closing the intervals X and Y . The arrows inside the domain (upper and left panels) indicate the points at which ψ acquires a discontinuity. The arrows outside the domain (right) indicate the values of x and y which acquire more than one PSE, violating thereby the properties $\mathcal{P}1$ - $\mathcal{P}2$. A crossed circle indicates that the point does not belong to the PSE functions.

The theorem below generalizes this observation. Note that the term *limit point* in relation to a sequence is used in the general topological meaning, as the limit point for the set of elements in the sequence (i.e., a point whose every open neighborhood contains an element of the sequence).

THEOREM 3 Let ψ be a \mathcal{C} -function and satisfy $\mathcal{P}1$. Then, for any sequence $x_n \rightarrow x$, with x and all x_n in X , the set of limit points of $h(x_n)$ in Y is either empty or consists of the point $h(x)$ alone (in which case, $h(x_n) \rightarrow h(x)$). The analogous statement holds for a \mathcal{C} -function ψ satisfying $\mathcal{P}2$ and sequences $y_n \rightarrow y$ in Y .

Proof. Assume the contrary: for some sequence $x_n \rightarrow x$, $h(x_n)$ has in Y a limit point $y \neq h(x)$. Then, for some subsequence of x_n (without loss of generality, the sequence itself), $h(x_n) \rightarrow y$, and we have

$$\psi(x_n, h(x_n)) \leq \psi(x_n, h(x)).$$

By the continuity of ψ ,

$$\begin{aligned} \psi(x_n, h(x_n)) &\rightarrow \psi(x, y), \\ \psi(x_n, h(x)) &\rightarrow \psi(x, h(x)), \end{aligned}$$

whence

$$\psi(x, y) \leq \psi(x, h(x)).$$

But this contradicts the definition of h . ■

For the next two corollaries, recall that a set A is called *sequentially compact* (or *Bolzano-Weierstrass*) if every infinite sequence in A has a limit point in A . If $B \subset A$, we define the *outer boundary* of B in A as

$$\{a \in A \setminus B : a \text{ is a limit point for } B\}.$$

COROLLARY 1 (TO THEOREM 3) If $Y \subset Y'$ and Y' is sequentially compact, then for every sequence $x_n \rightarrow x$ such that $h(x_n) \not\rightarrow h(x)$, all limit points of $h(x_n)$ belong to the outer boundary of Y in Y' . (Analogously for X and $g(y_n)$.)

This corollary describes the situation we see in Figs. 6, 8, 9, with Y an open interval of reals, and Y' its closure.

The next corollary explains the failure of constructing a \mathcal{C} -function with discontinuous PSE functions in Fig. 10.

COROLLARY 2 (TO THEOREM 3) If Y (or X) is sequentially compact, then h (respectively, g) is continuous. In particular, if X and Y are compact subspaces of \mathbb{R}^n , both h and g are continuous.

All of this shows that adding the requirement that a \mathcal{C} -function have continuous PSE functions does not overly restrict the class of \mathcal{C} -functions (subject to $\mathcal{P}1$ - $\mathcal{P}2$). With this requirement in place we can formulate a simple topological criterion for the implication $\text{Const} \rightarrow \text{RegMin}$.

THEOREM 4 Let ψ be a \mathcal{C} -function which satisfies $\mathcal{P}1$ - $\mathcal{P}2$ with continuous PSE functions h, g , and let $\omega_h(x) \equiv c$ (or $\omega_g(y) \equiv c$). Then ψ satisfies $\mathcal{P}3$ if and only if $h(X)$ is open in Y (respectively, $g(Y)$ is open in X).

Proof. The necessity (only if) is obvious: if $\mathcal{P}3$ holds, $h(X) = Y$ is open in Y . To prove the sufficiency, denote by g^* the restriction $g|_{h(X)}$ of g (obviously, nonempty). By Lemma 2, $g^* \equiv h^{-1}$. We prove that if $h(X)$ is open, $Y \setminus h(X)$ is empty. Assume the contrary. Since $Y \setminus h(X)$ is closed in Y , and since it cannot also be open in Y (for Y is connected), there is a point $y^* \in Y \setminus h(X)$ such that its every open neighborhood contains points of $h(X)$. This implies the existence of a sequence $y_n \rightarrow y^*$ with all $y_n \in h(X)$. Since g is continuous, the sequence $g(y_n) = g^*(y_n)$ should converge to $x^* = g(y^*)$. But h is continuous too, whence $y_n = h(g^*(y_n)) \rightarrow h(g(y^*)) \in h(X)$. We have then

$$y_n \rightarrow y^* \in Y \setminus h(X)$$

and

$$h(g^*(y_n)) = y_n \rightarrow h(g(y^*)) \in h(X),$$

a contradiction. ■

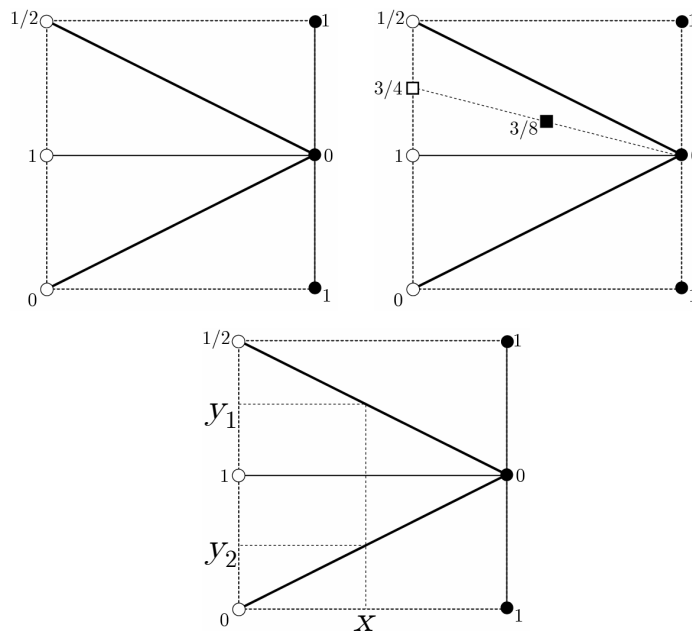


Figure 11. A scheme for constructing a continuous function ψ defined on the Cartesian product of a half-open (X) and open (Y) real intervals. A small-size number at a point indicates the value of ψ at this point (closed circle), or the limit value of ψ if the point is outside the domain (open circle). The right panel shows how to fill in the rest of the values. The limit values of ψ on an edge of the domain are computed by linear interpolations between the nearest points on the same edge with values marked (e.g., the open square is halfway between $\frac{1}{2}$ and 1, hence its value is $\frac{3}{4}$). By linearly interpolating between these limit points and the zero-valued point on the right (as shown by the dashed line) we find values for all interior points (e.g., the closed square is halfway between 0 and $\frac{3}{4}$, hence its value is $\frac{3}{8}$). Bottom panel: The segment of the heavy line between the two zero-valued points graphs the PSE function $h(x)$; the entire, two-segment, heavy line is the graph of the PSE function $g(y)$. Regular Minimality is violated because $x = g(y_1)$ but $h(x) = y_2 \neq y_1$.

The example in Fig. 11 shows that the openness of $h(X)$ is critical for this result: if $h(X)$ is not open in Y , then a \mathcal{C} -function with continuous PSE functions and a constant minimum level function ω_h , may violate Regular Minimality.

The following corollary is of interest due to the prominence of stimulus spaces representable by open connected regions of \mathbb{R}^n in initial formulations of the Regular Minimality law (Dzhafarov, 2002, 2003).

COROLLARY 3 (TO THEOREM 4) If X and Y are open connected regions of \mathbb{R}^n , and ψ a continuous function with continuous PSE functions, then

$$\omega_h(x) \equiv c \text{ (or } \omega_g(y) \equiv c) \implies \mathcal{P3}.$$

Proof. By Brouwer's invariance of domain theorem (see, e.g., Hocking & Young, 1961, pp. 277-278) every injective continuous function from a subset of \mathbb{R}^n into \mathbb{R}^n is open (maps open sets onto open sets). Hence $h(X)$ is open in Y , and Theorem 4 applies. ■

3. From Regular Minimality to Constancy

Here, we discuss properties of discrimination functions ψ which enable the implication RegMin→Const. It is easy to see that the topological properties invoked in the preceding section are not sufficient. In fact, as illustrated by Fig. 12, if X and Y are intervals of reals (no matter if closed, open, or half-open), then any homeomorphism (which in this case means any continuous bijection) $h : X \rightarrow Y$ and any continuous function $\omega_h : X \rightarrow [0, 1[$ can serve as, respectively, the PSE function and the minimum level function for a continuous function $\psi : X \times Y \rightarrow [0, 1]$.

This observation can be easily generalized to all topological X and Y which allow for at least one continuous discrimination function subject to Regular Minimality with a continuous PSE function. Note that a continuous PSE function under Regular Minimality is a homeomorphism $X \rightarrow Y$ (a continuous bijection with a continuous inverse).

LEMMA 3 Let X, Y be topological spaces, and let \mathcal{U} be the class of continuous functions $X \times Y \rightarrow [0, 1]$ subject to Regular Minimality with continuous PSE functions. If \mathcal{U} is nonempty, then for any homeomorphism $h : X \rightarrow Y$ and any continuous $\omega_h : X \rightarrow [0, 1]$ there is a function $\psi \in \mathcal{U}$ with h and ω_h as its PSE function and minimum level function, respectively.

Proof. Let $\psi^* \in \mathcal{U}$, with a homeomorphic $h^* : X \rightarrow Y$ as its PSE function. Denote $g^* \equiv (h^*)^{-1}$ and define

$$\begin{aligned} \delta_X(x, x') &= \psi^*(x, h^*(x')) - \psi^*(x, h^*(x)), \\ \delta_Y(y, y') &= \psi^*(g^*(y'), y) - \psi^*(g^*(y'), y). \end{aligned}$$

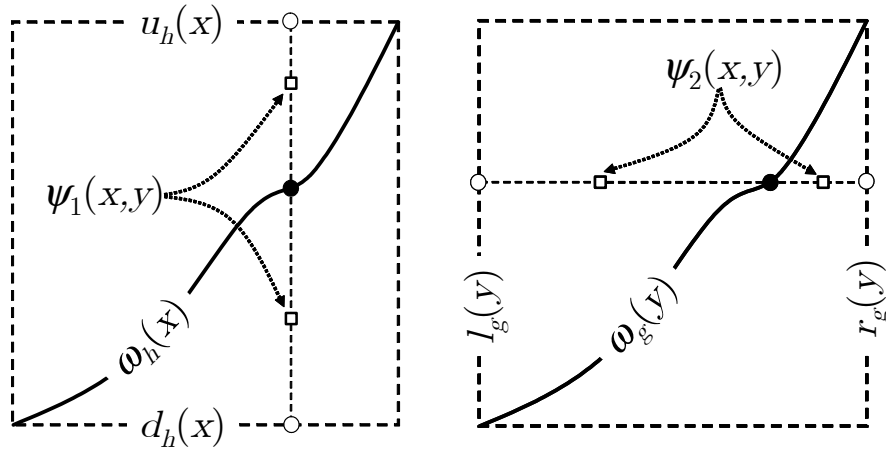


Figure 12. Construction of a continuous function ψ with a given continuous PSE curve (graphed by the solid curve) and a given continuous minimum level function, equivalently written as $\omega_h(x)$ or $\omega_g(y)$. X and Y are intervals of reals (shown to be open, but this is immaterial), and $\psi(x, y) = \max\{\psi_1(x, y), \psi_2(x, y)\}$, with the construction of ψ_1 and ψ_2 shown in the left and right panels, respectively. Left panel: Choosing arbitrary continuous $u_h(x) > \omega_h(x)$, $d_h(x) > \omega_h(x)$, and assigning them to the upper and lower boundaries, $\psi_1(x, y)$ is computed by linear interpolation between the values $u_h(x)$ and $\omega_h(x)$, or between $\omega_h(x)$ and $d_h(x)$, according as y is above or below the PSE curve. Right panel: With arbitrary continuous $l_g(y) > \omega_g(y)$, $r_g(y) > \omega_g(y)$ assigned to the left and right boundaries, $\psi_2(x, y)$ is computed by linear interpolation between the values $l_g(y)$ and $\omega_g(y)$, or between $\omega_g(y)$ and $r_g(y)$, according as y is to the left or to the right of the PSE curve.

Clearly, $\delta_X : X \times X \rightarrow [0, 1]$ and $\delta_Y : Y \times Y \rightarrow [0, 1]$ are continuous functions, and

$$\begin{aligned}\delta_X(x, x') &= 0 \iff x = x', \\ \delta_Y(y, y') &= 0 \iff y = y'.\end{aligned}$$

Let a homeomorphism $h(x)$ and a continuous $\omega_h(x)$ be given. Denote $g \equiv h^{-1}$ (a homeomorphism), put $\omega_g(y) = \omega_h(g(y))$ (a continuous function), and define

$$\begin{aligned}\psi_1(x, y) &= \min\{\omega_h(x) + \delta_Y(h(x), y), 1\} \\ \psi_2(x, y) &= \min\{\omega_g(y) + \delta_X(g(y), x), 1\} \\ \psi(x, y) &= \max\{\psi_1(x, y), \psi_2(x, y)\}.\end{aligned}$$

It is easy to verify that ψ satisfies the statement of the lemma. ■

In fact, the assumption of the lemma can be weakened: it would suffice to posit the existence of at least one continuous function $\psi^*(x, y)$ satisfying $\mathcal{P}1$ with a continuous surjective PSE function $h^*(x)$, and at least one $\psi^{**}(x, y)$ satisfying $\mathcal{P}2$ with a continuous surjective PSE function $g^{**}(x)$.

The lemma shows that the possible generalizations of (7) mentioned in Introduction in relation to $\text{RegMin} \rightarrow \text{Const}$ should be analytic rather than topological. It turns out that analytic considerations can be extracted from a topological context in a natural way if one replaces the connectedness property in the definition of a \mathcal{C} -function with the stronger property of *arc-connectedness*.

3.1. Arcwise parametrization of stimuli

A Hausdorff space A is arc-connected if (and only if), for any distinct $p, q \in A$ one can find an injective continuous map (an *arc*) $a : [0, 1] \rightarrow A$ with $a(0) = p$ and $a(1) = q$. An arc is a homeomorphism from its domain $[0, 1]$ onto its image $a([0, 1])$. It is convenient to present a as a_p^q , to indicate its endpoints and to distinguish it from points in A . So we will speak of arcs a_p^q with points $a(t)$, $t \in [0, 1]$. To distinguish an arc as a mapping, $a_p^q : [0, 1] \rightarrow A$, from its image $a_p^q([0, 1])$ in A , we denote the image $[a_p^q]$. Clearly, different arcs a_p^q and b_p^q may have the same image, $[a_p^q] = [b_p^q]$.

We assume now that

- (\mathcal{A}) X and Y are Hausdorff, first countable, *arc-connected* topological spaces, and ψ is continuous with respect to the product topology on $X \times Y$.

Such a function ψ will be referred to as an \mathcal{A} -function.⁵

Given any two arcs $x_u^{u'} : [0, 1] \rightarrow X$ and $y_v^{v'} : [0, 1] \rightarrow Y$ (with endpoints u, u' and v, v' , respectively), the function

$$\varphi(s, t) = \psi(x(s), y(t)) \tag{8}$$

is called an *arc-parametrized facet* (*AP-facet*, for short) of ψ . Figure 13 provides a schematic illustration. Since ψ is continuous, $\varphi(s, t)$ is continuous (hence uniformly continuous) on $[0, 1] \times [0, 1]$. Clearly, an AP-facet of an \mathcal{A} -function is an \mathcal{A} -function.

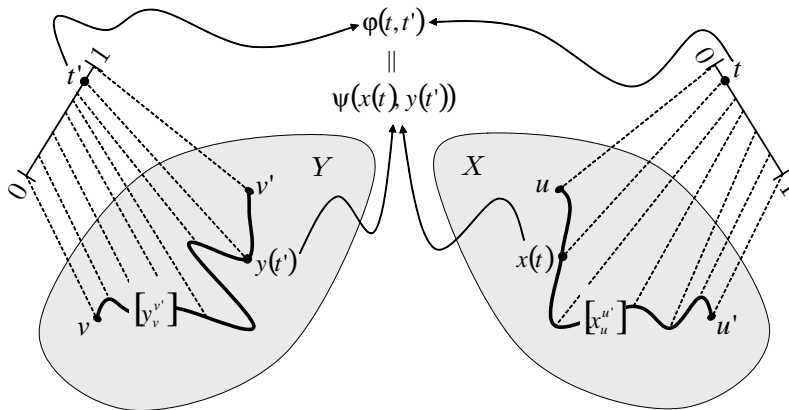


Figure 13. An illustration for the notion of an arc-parametrized facet φ of ψ . Explanations for the symbols are given in the text.

We should strictly distinguish between an AP-facet φ of ψ and a restriction of ψ to arc images $[x_u^{u'}] \times [y_v^{v'}] \subset X \times Y$. Every pair of arc images can be associated with an infinity of possible parametrizations

$$x_u^{u'} : [0, 1] \rightarrow [x_u^{u'}] \quad \text{and} \quad y_v^{v'} : [0, 1] \rightarrow [y_v^{v'}]. \tag{9}$$

⁵The first countability is not in fact utilized in the subsequent discussion. It is convenient, however, to keep it in place to ensure that all \mathcal{A} -functions are \mathcal{C} -functions.

Let an \mathcal{A} -function ψ satisfy Regular Minimality with continuous PSE functions h and $g = h^{-1}$ (more compactly, with a homeomorphic PSE function h). Then, for any arc image $[x_u^{u'}]$ in X ,

$$h\left([x_u^{u'}]\right) = [y_v^{v'}]$$

is an arc image in Y . Such $[x_u^{u'}]$ and $[y_v^{v'}]$ will be referred to as *PSE-corresponding* arc images. For any parametrizations (9) of these two arc images, we have a bijective correspondence $\pi : [0, 1] \rightarrow [0, 1]$ between the domains of the two arcs, where

$$\pi \equiv \left(y_v^{v'}\right)^{-1} \circ h \circ x_u^{u'}. \quad (10)$$

The meaning of π is given by the following lemma.

LEMMA 4 Let ψ be an \mathcal{A} -function subject to Regular Minimality, with a homeomorphic PSE function h . Then any AP-facet φ of ψ defined by arcs $x_u^{u'}$ and $y_v^{v'}$ with PSE-corresponding images is subject to Regular Minimality, with a homeomorphic PSE function π given by (10).

Proof. As the functions $y_v^{v'}$, h , and $x_u^{u'}$ in (10) are homeomorphisms, π is a homeomorphism. It also follows from (10) and the bijectivity of $y_v^{v'}$ that, for any $t \neq \pi(s)$ in $[0, 1]$,

$$y(t) \neq y(\pi(s)) = h(x(s)).$$

It follows then from (8) that

$$\varphi(s, \pi(s)) = \psi(x(s), h(x(s))) < \psi(x(s), y(t)) = \varphi(s, t).$$

Analogously, if $s \neq \pi^{-1}(t)$,

$$\varphi(\pi^{-1}(t), t) = \psi(h^{-1}(y(t)), y(t)) < \psi(x(s), y(t)) = \varphi(s, t).$$

This establishes that φ satisfies Regular Minimality with π as its PSE function. ■

Due to its importance we state the PSE property of π separately: for any $t, s \in [0, 1]$,

$$t \neq \pi(s) \implies \varphi(s, t) > \max\{\varphi(s, \pi(s)), \varphi(\pi^{-1}(t), t)\} \quad (11)$$

The simplest parametrizations of two PSE-corresponding arcs are, of course, those with

$$y_v^{v'} \equiv h \circ x_u^{u'} \iff x_u^{u'} \equiv h^{-1} \circ y_v^{v'}. \quad (12)$$

With such a choice π is the identity function $[0, 1] \rightarrow [0, 1]$, and (11) assumes the *canonical form*

$$t \neq s \implies \varphi(s, t) > \max\{\varphi(s, s), \varphi(t, t)\}. \quad (13)$$

3.2. Well-behaved discrimination functions

We use the following notation for finite differences of the first and second order. For any $s, s', t, t' \in [0, 1]$,

$$\begin{aligned}\Delta_{s'}^1 \varphi(s, t) &= \varphi(s', t) - \varphi(s, t), \\ \Delta_{t'}^2 \varphi(s, t) &= \varphi(s, t') - \varphi(s, t),\end{aligned}\tag{14}$$

where the superscript refers to the position of the argument changed. Analogously,

$$\begin{aligned}\Delta_{(s', t')}^{12} \varphi(s, t) &= \Delta_{s'}^1 \Delta_{t'}^2 \varphi(s, t) = \Delta_{t'}^2 \Delta_{s'}^1 \varphi(s, t) \\ &= \varphi(s', t') - \varphi(s', t) - \varphi(s, t') + \varphi(s, t).\end{aligned}\tag{15}$$

It is easy to verify that

$$\varphi(s', t') - \varphi(s, t) = \begin{cases} \Delta_{s'}^1 \varphi(s, t) + \Delta_{t'}^2 \varphi(s, t) + \Delta_{(s', t')}^{12} \varphi(s, t) \\ -\Delta_s^1 \varphi(s', t') - \Delta_t^2 \varphi(s', t') - \Delta_{(s', t')}^{12} \varphi(s, t) \end{cases}.\tag{16}$$

Another notation convention: we use double arrows $(s', t') \rightrightarrows (s, t)$ to indicate that s' and t' approach, respectively, s and t from the same side. Specifically:

$$\begin{aligned}(s', t') \rightrightarrows (s, t) + & \text{ means } s' \rightarrow s+ \text{ and } t' \rightarrow t+, \\ (s', t') \rightrightarrows (s, t) - & \text{ means } s' \rightarrow s- \text{ and } t' \rightarrow t-, \\ (s', t') \rightrightarrows (s, t) \pm & \text{ means one of the two: } \begin{aligned} & (s', t') \rightrightarrows (s, t) + \\ & (s', t') \rightrightarrows (s, t) - \end{aligned} \\ (s', t') \rightrightarrows (s, t) & \text{ means } s' \rightarrow s \text{ and } t' \rightarrow t \text{ and } (s' - s)(t' - t) \geq 0\end{aligned}\tag{17}$$

Given an \mathcal{A} -function ψ and a pair of arc images, $[x_u^{u'}]$ and $[y_v^{v'}]$, we say that the restriction $\psi \left[[x_u^{u'}] \times [y_v^{v'}] \right]$ of ψ is *well-behaved on* $[x_u^{u'}]$ if, for some parametrizations $x_u^{u'} : [0, 1] \rightarrow [x_u^{u'}]$ and $y_v^{v'} : [0, 1] \rightarrow [y_v^{v'}]$, the resulting AP-facet φ of ψ has the following properties:

($\mathcal{R}1$) for all $(s, t) \in [0, 1] \times [0, 1]$ except for an at most denumerable set,

$$\limsup_{(s', t') \rightrightarrows (s, t)} \left| \frac{\Delta_{(s', t')}^{12} \varphi(s, t)}{s' - s} \right| < \infty;\tag{18}$$

($\mathcal{R}2$) for almost all $s \in [0, 1]$ and almost all $t \in [0, 1]$,

$$\lim_{(s', t') \rightrightarrows (s, t) \pm} \frac{\Delta_{(s', t')}^{12} \varphi(s, t)}{s' - s} = 0,\tag{19}$$

where the choice of $+$ or $-$ may depend on (s, t) .

The definition of a restriction $\psi \left[[x_u^{u'}] \times [y_v^{v'}] \right]$ well-behaved on $[y_v^{v'}]$ is obtained by replacing the quotient in (18) and (19) with

$$\frac{\Delta_{(s', t')}^{12} \varphi(s, t)}{t' - t}.$$

The motivation for this definition can be seen by presenting the quotient in (18) and (19) as

$$\frac{\Delta_{(s',t')}^{12}\varphi(s,t)}{s'-s} = \frac{\varphi(s',t') - \varphi(s,t')}{s'-s} - \frac{\varphi(s',t) - \varphi(s,t)}{s'-s}. \quad (20)$$

According to Dzhafarov (2003), $\varphi(s,t)$ is “near-smooth” if it has unilateral derivatives in s (and t) which are continuous in t (respectively, s). If so, one can easily see in (20) that (19) holds for both $(s',t') \rightrightarrows (s,t) +$ and $(s',t') \rightrightarrows (s,t) -$ at every (s,t) , and (18) is satisfied ipso facto. The continuous differentiability mentioned in relation to (7) is a special case of near-smoothness. At the same time, the generalization provided by $\mathcal{R}1$ - $\mathcal{R}2$ is considerable: the limits for the right-hand ratios in (20) need not exist or be finite even when (19) holds, the quotient in the latter may be nonzero on a product of two sets of measure zero, and it can grow infinitely large in absolute value on a denumerable set. Also, the definition says nothing about the behavior of the quotient when s' and t' approach, respectively, s and t from different sides.

Given a parametrization $a_p^q : [0, 1] \rightarrow A$ of an arc image $[a_p^q]$, a *reparametrization* of $[a_p^q]$ is

$$b_p^q \equiv a_p^q \circ f \quad (21)$$

where f is a strictly increasing continuous mapping of $[0, 1]$ onto $[0, 1]$ (hence a homeomorphism). Clearly, $[a_p^q] = [b_p^q]$.

The following observation is obvious but important.

LEMMA 5 Let a restriction $\psi \left| \left[x_u^{u'} \right] \times \left[y_v^{v'} \right] \right.$ of ψ be well-behaved on $\left[x_u^{u'} \right]$ (or on $\left[y_v^{v'} \right]$), and let the parametrizations $x_u^{u'} : [0, 1] \rightarrow \left[x_u^{u'} \right]$ and $y_v^{v'} : [0, 1] \rightarrow \left[y_v^{v'} \right]$ satisfy $\mathcal{R}1$ - $\mathcal{R}2$. Then $\mathcal{R}1$ - $\mathcal{R}2$ are satisfied under all reparametrizations of $\left[y_v^{v'} \right]$ (respectively, $\left[x_u^{u'} \right]$).

Proof. A reparametrization of $\left[y_v^{v'} \right]$ amounts to choosing an increasing homeomorphism $f : [0, 1] \rightarrow [0, 1]$ and replacing t with $f(\tau)$ in $\mathcal{R}1$ - $\mathcal{R}2$. Obviously, $\varphi(s,t) = \varphi(s, f(\tau))$ and $f(\tau') \rightarrow f(\tau) \pm$ is equivalent to $\tau' \rightarrow \tau \pm$. ■

We are now in a position to formulate the analytic characterization of ψ to be related to the implication $\text{RegMin} \rightarrow \text{Const}$.

Let $A(\psi)$ denote the set of all ordered products of arc images $\left[x_u^{u'} \right] \times \left[y_v^{v'} \right]$ such that $\psi \left| \left[x_u^{u'} \right] \times \left[y_v^{v'} \right] \right.$ is well-behaved on at least one of the two arc images, $\left[x_u^{u'} \right]$ or $\left[y_v^{v'} \right]$. Let h be a homeomorphism $X \rightarrow Y$. We say that ψ is *well-behaved with respect to h* if, for every $u, u' \in X$, there is an arc image $\left[x_u^{u'} \right] \subset X$ such that

$$\left[x_u^{u'} \right] \times h \left(\left[x_u^{u'} \right] \right) \in A(\psi).$$

As h maps X onto Y , it follows that for every $v, v' \in Y$ there is an arc image $\left[y_v^{v'} \right] \subset Y$ such that

$$h^{-1} \left(\left[y_v^{v'} \right] \right) \times \left[y_v^{v'} \right] \in A(\psi).$$

We are, of course, interested in ψ well-behaved with respect to the homeomorphic PSE function h , provided ψ satisfies Regular Minimality.

THEOREM 5 Let ψ be an \mathcal{A} -function subject to Regular Minimality, with a homeomorphic PSE function h , and let ψ be well-behaved with respect to h . Then the minimum level function $\omega_h(x) = \psi(x, h(x))$ is constant.

Proof. (The proof makes use of Lemma 6, stated immediately after the theorem.) In accordance with the definition of ψ well-behaved with respect to h , for any $u, u' \in X$ we can find $[x_u^{u'}] \times h([x_u^{u'}]) \in A(\psi)$. Without loss of generality, let $\psi|_{[x_u^{u'}] \times h([x_u^{u'}])}$ be well-behaved on $[x_u^{u'}]$, and let the AP-facet φ defined by parametrizations

$$x_u^{u'} : [0, 1] \rightarrow [x_u^{u'}] \quad \text{and} \quad y_v^{v'} : [0, 1] \rightarrow h([x_u^{u'}])$$

satisfy $\mathcal{R}1$ - $\mathcal{R}2$. By Lemma 5, the parametrization $y_v^{v'}$ can be chosen arbitrarily. We choose

$$y_v^{v'} \equiv h \circ x_u^{u'},$$

so that the AP-facet φ satisfies (13). Then, for $t' \neq t$,

$$\begin{aligned} \Delta_{t'}^1 \varphi(t, t) &> 0, & \Delta_{t'}^2 \varphi(t, t) &> 0, \\ \Delta_t^1 \varphi(t', t') &> 0, & \Delta_t^2 \varphi(t', t') &> 0. \end{aligned}$$

Applying (16) to $s = t$ and $s' = t' \neq t$, we get

$$\varphi(t', t') - \varphi(t, t) = \begin{cases} \Delta_{t'}^1 \varphi(t, t) + \Delta_{t'}^2 \varphi(t, t) + \Delta_{(t', t')}^{12} \varphi(t, t) > \Delta_{(t', t')}^{12} \varphi(t, t) \\ -\Delta_t^1 \varphi(t', t') - \Delta_t^2 \varphi(t', t') - \Delta_{(t', t')}^{12} \varphi(t, t) < -\Delta_{(t', t')}^{12} \varphi(t, t) \end{cases},$$

whence

$$|\varphi(t', t') - \varphi(t, t)| < -\Delta_{(t', t')}^{12} \varphi(t, t).$$

The properties $\mathcal{R}1$ - $\mathcal{R}2$ apply to $s = t$ and $s' = t' \neq t$, with $(t', t') \Rightarrow (t, t)$ being equivalent to $t' \rightarrow t$, and $(t', t') \Rightarrow (t, t) \pm$ to $t' \rightarrow t \pm$ (with the same choice of $+$ or $-$). It follows then from $\mathcal{R}1$ that

$$\limsup_{t' \rightarrow t} \left| \frac{\varphi(t', t') - \varphi(t, t)}{t' - t} \right| < \infty$$

except on an at most denumerable set, and it follows from $\mathcal{R}2$ that for almost every $t \in [0, 1]$,

$$\frac{d\varphi(t, t)}{dt \pm} = \lim_{t' \rightarrow t \pm} \frac{\varphi(t', t') - \varphi(t, t)}{t' - t} = 0.$$

By Lemma 6, we conclude that

$$\varphi(t, t) \equiv \text{const.}$$

In particular, on recalling that φ is defined by arcs $x_u^{u'}$ and $h \circ x_u^{u'}$,

$$\psi(u, h(u)) = \varphi(0, 0) = \varphi(1, 1) = \psi(u', h(u')).$$

As this is true for all $u, u' \in X$, the theorem is proved. ■

LEMMA 6 (FOR THEOREM 3) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and suppose that for almost every $t \in [0, 1]$, either $f'_+(t) = 0$ or $f'_-(t) = 0$. If, in addition,

$$\limsup_{t' \rightarrow t} \left| \frac{f(t') - f(t)}{t' - t} \right| < \infty$$

for all t except for an at most denumerable set, then f is constant.

Proof. This is a corollary to Theorem 2 in Miller and Vyborny (1986). ■

4. Conclusion

We have established several relations between the properties of Regular Minimality and the constancy of the minimum level functions for a discrimination function ψ . To highlight some of them,

1. (Const \rightarrow RegMin type, Theorems 1 and 2) ψ with well-defined PSE functions (i.e., satisfying $\mathcal{P}1$ - $\mathcal{P}2$) complies with Regular Minimality if both its minimum level functions are constant, or if one of them is constant and the corresponding PSE function is onto;
2. (Const \rightarrow RegMin type, Theorem 4) for a continuous function ψ with continuous PSE functions (on Hausdorff, first countable, connected stimulus spaces), if the range of a PSE function is open then its constancy implies Regular Minimality;
3. (RegMin \rightarrow Const type, Theorem 5) for stimulus spaces which are also arc-connected, the Regular Minimality property of ψ implies the constancy of its minimum level function if ψ is well-behaved with respect to its homeomorphic PSE function h (Section 3.2), the well-behavedness being a property generalizing continuous differentiability of $\psi(x, y)$ with real-valued x and y .

These and accompanying results are simple, but they are not immediately obvious and have not been previously stated.

Some of the intermediate observations and constructions presented in this paper may be of interest in their own right, outside the context of the relationship between Regular Minimality and the constancy of minimum level functions. Thus,

- (a) Theorem 3 and its corollaries clarify the relationship between the continuity of ψ and the continuity of its PSE functions;
- (b) Lemma 3 demonstrates the existence, for a broad class of topological spaces, of a continuous ψ subject to Regular Minimality with any continuous PSE function and any continuous minimum level function;
- (c) most importantly, Section 3 introduces the representation of ψ by its “arc-parametrized facets,” a construction which allows one to discuss analytic properties of a function whose domain is characterized in entirely topological terms (Hausdorff, first countable, arc-connected stimulus spaces).

Among other uses the latter construction opens the way for a sweeping generalization of Dzhafarov’s (2003) analysis of Thurstonian-type modeling of discrimination probabilities (which was confined to stimuli representable in \mathbb{R}^n). This issue will be dealt with in another paper.

With the topological properties such as Hausdorff topology and first countability playing a critical role in our development, the question arises as to where this topology is derived from. The obvious answer seems to be: from physical descriptions of the stimuli. If a stimulus, such as a point-size spot of light uniformly moving from a fixed position to the right or to the left, is described by an interval of reals (say, velocity values), the Euclidean topology of real numbers seems to determine the topology of the stimulus space. The problem with this approach is that a mathematical set does not uniquely determine its topology, and that one and the same set of stimuli allows for multiple mathematical sets to describe it. To use a simplistic demonstration, the space of visual motions just mentioned may be represented by $V = [-1, 1]$, with zero corresponding to rest and positive/negative values representing the rightward/leftward velocities. It would be also legitimate, however, to measure the same motions by the time it would take them to cover a unit distance, taken with negative sign if the motion is leftward. The representing set $T =]-\infty, -1] \cup [1, \infty[$ for this description (with ∞ indicating the state of rest), differs topologically from V if in both cases the topologies are induced by one of the conventional topologies of extended reals. On the other hand, nothing prevents one from imposing on T the Euclidean topology of V , by considering a set open in T if and only if the set of the reciprocals of its elements is open in V in the usual sense. We see that the choice of a topological structure based on physical description is far from being unambiguous.

A different approach to the problem of where stimulus sets acquire their structure (including topology) is introduced in Dzhafarov and Colonius (2005, 2007). The approach is called “purely psychological” (as opposed to “psycho-physical”): all structural properties of stimulus sets X, Y are derived from the discrimination function ψ defined on $X \times Y$, with physical descriptions serving as mere labels. With minor expository variations and omitting details, the topological construction in Dzhafarov & Colonius (2007) is as follows.

Our first assumption is that ψ satisfies Regular Minimality ($\mathcal{P}1$ - $\mathcal{P}3$), with a PSE function $h : X \rightarrow Y$. Then the *psychometric increments* of the first and second kind, defined as

$$\begin{aligned} \Psi^{(1)}(x, x') &= \psi(x, h(x')) - \psi(x, h(x)), \\ \Psi^{(2)}(x, x') &= \psi(x', h(x)) - \psi(x, h(x)), \end{aligned} \tag{22}$$

are positive for $x \neq x'$ and vanish at $x = x'$. We choose one of them, say $\Psi^{(1)}$, and make our second assumption: as $\Psi^{(1)}(x_1, x'_1) \rightarrow 0$ and $\Psi^{(1)}(x_2, x'_2) \rightarrow 0$,

$$\begin{aligned} \Psi^{(1)}(x'_1, x'_2) - \Psi^{(1)}(x_1, x_2) &\rightarrow 0, \\ \Psi^{(2)}(x'_1, x'_2) - \Psi^{(2)}(x_1, x_2) &\rightarrow 0. \end{aligned} \tag{23}$$

(Note that none of the four points x_1, x'_1, x_2, x'_2 is assumed to be fixed here.) The topology on X is introduced by taking the sets

$$B(x, \varepsilon) = \left\{ x' \in X : \Psi^{(1)}(x, x') < \varepsilon \right\}$$

for all $x \in X$ and $\varepsilon > 0$ as a topological base. This means that open sets in X are created by taking all possible unions of sets $B(x, \varepsilon)$. (Replacing $\Psi^{(1)}(x, x')$ with $\Psi^{(1)}(x', x)$, $\Psi^{(2)}(x, x')$, or $\Psi^{(2)}(x', x)$ creates the same topology.) The topology on Y can be imposed by simply positing that a subset $Y' \subset Y$ is open in Y if and only if $h^{-1}(Y')$ is open in X . This automatically makes h a homeomorphism. It can be proved now that the function ψ on which this construction is based is continuous in the product topology of X and Y . (In fact, the assumption (23) allows us to impose on X and Y a *uniformity*, a richer structure than topology, and to prove that ψ is uniformly continuous.)

It is easy to see that X and Y with the topology thus constructed are always Hausdorff and first countable. The notion of an arc being well-defined, one can focus on the subclass of X, Y which are arc-connected and effect the entire construction of Section 3, with Theorem 5 applicable and valid. A limitation of the “purely psychological” approach lies in the fact that it is based on the law of Regular Minimality as its cornerstone. As a result, one cannot pose within its framework any of the questions discussed in Section 2. Whether this approach can be generalized to discrimination functions subject to $\mathcal{P}1$ - $\mathcal{P}2$ but not necessarily $\mathcal{P}3$ remains to be seen.

Acknowledgement. This research has been supported by NSF grant SES 0620446 and AFOSR grant FA9550-06-1-0288 to Purdue University, as well as by the Commission of the European Communities, Marie Curie Excellence Grant MEXT-CT-2004-014203 to University of Jyväskylä. The authors are grateful to Hans Colonius, Reinhard Suck, and an anonymous reviewer for many helpful suggestions.

References

- Dzhafarov, E.N. (2002). Multidimensional Fechnerian scaling: Pairwise comparisons, regular minimality, and nonconstant self-similarity. *Journal of Mathematical Psychology*, **46**, 583-608.
- Dzhafarov, E.N. (2003). Thurstonian-type representations for “same-different” discriminations: Deterministic decisions and independent images. *Journal of Mathematical Psychology*, *47*, 208-228.
- Dzhafarov, E.N. (2006). On the Law of Regular Minimality: Reply to Ennis. *Journal of Mathematical Psychology*, *50*, 74-93.
- Dzhafarov, E.N., & Colonius, H. (2005). Psychophysics without physics: A purely psychological theory of Fechnerian Scaling in continuous stimulus spaces. *Journal of Mathematical Psychology*, *49*, 1-50.
- Dzhafarov, E.N., & Colonius, H. (2006). Regular Minimality: A fundamental law of discrimination. In H. Colonius & E.N. Dzhafarov (Eds.), *Measurement and Representation of Sensations* (pp. 1-46). Mahwah, NJ: Erlbaum.
- Dzhafarov, E.N., & Colonius, H. (2007). Dissimilarity Cumulation theory and subjective metrics. *Journal of Mathematical Psychology*, *51*, 290-304.

Ennis, D. M. (2006). Sources and influence of perceptual variance: Comment on Dzhafarov's Regular Minimality Principle. *Journal of Mathematical Psychology*, 50, 66–73.

Hocking, J.H. & Young, G.S. (1961). *Topology*. Reading, MA: Addison-Wesley.

Luce, R. D., & Galanter, E. (1963). Discrimination. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of Mathematical Psychology*, v. 1 (pp. 103–189). New York: Wiley.

Miller, A.D. & Vyborny, R. (1986). Some remarks on functions with one-sided derivatives. *The American Mathematical Monthly*, 93, 471–475.