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Selective Influence and Response Time Cumulative Distribution Functions in Serial-Parallel Task Networks

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We analyze sets of mental processes, some of which are concurrent and some of which are sequential, under the assumption that the processes are partially ordered, that is, arranged in a directed acyclic network. Information about the process arrangement can be discovered by examining the effects on response time of selectively influencing process durations. Previous work has mainly focused on analyses of mean response times. Here we consider analyses based on cumulative distribution functions, for one of the major classes of directed acyclic networks, serial-parallel networks. When two processes are selectively influenced, patterns in the cumulative distribution functions can be used to test whether the processes are sequential or concurrent and whether the task network has AND gates or OR gates. Cumulative distribution functions are potentially more informative than means, and some previous results for means are shown to follow from our results for cumulative distribution functions.

Considerable evidence indicates that the mental processes involved in some information processing tasks are neither entirely serial nor entirely parallel. Many alternative forms of processing are conceivable; one of the plausible and useful forms is a directed acyclic network, such as the one in Fig. 1. The network represents a combination of sequential and concurrent processing. The stimulus is presented at o, and processes x and z begin immediately and concurrently. Process y begins as soon as x is finished. The response is made at r as soon as both y and z are finished, if there is an AND gate, or as soon as either y or z is finished, if there is an OR gate.

There are many examples of tasks for which good accounts of response time data have been given by directed acyclic networks (although not always by that name).

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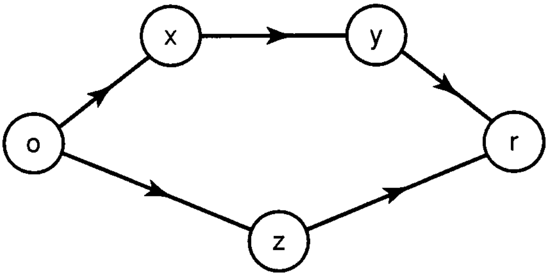

FIG. 1. A directed acyclic network. This is an example of a serial-parallel network.

The tasks are primarily dual tasks, in which a subject is presented with two stimuli and makes a response to each. Studies include Davis (1957), De Jong (1993), Ehrenstein, Schweickert, Choi, and Proctor (1997), Johnston, McCann, and Remington (1995), McCann and Johnston (1992), Osman and Moore (1993), Pashler and Johnston (1989), Ruthruff, Miller, and Lachmann (1995), Schweickert (1983), Van Selst and Jolicoeur (1994), and Welford (1952). In a recent model by Meyer and Kieras (1997a, 1997b) a set of directed acyclic networks is sometimes available, from which one is selected for a given trial. For reviews, see Pashler (1998) and Schweickert and Boggs (1984).

The arrangement of the processes in a directed acyclic network can be discovered if experimental factors are available that selectively influence the durations of individual processes. The technique of selective influence was pioneered by Sternberg (1969) in his Additive Factor Method for tasks in which all the processes are in series. He noted that if one experimental factor prolongs one process, while another factor prolongs another process, then the combined effect on mean response time of prolonging both processes will be the sum of the effects of prolonging them individually.

In later developments, the technique of selectively influencing processes was extended to more complicated architectures than serial processing. For directed acyclic networks, the architecture of interest here, Schweickert (1978) showed how effects of selectively influencing processes could be used to discover details about the process arrangement, including whether a pair of processes is sequential or concurrent. The problem of inferring the arrangement of mental processes from measurable quantities has received concerted attention from many people over an extended period of time. It is not possible to summarize it all here, but recent surveys can be found in Coles, Smid, Scheffers, and Otten (1995), Luce (1986), Massaro and Cowan (1993), Schweickert (1992), Townsend (1990a), and Townsend and Ashby (1983).

For the most part, effects of selectively influencing processes have been evaluated in terms of mean response times. But a mean is only a summary, and complete information about a distribution is in its cumulative distribution function. Cumulative distribution functions have been investigated in a variety of ways; for examples, see Balakrishnan (1994), Colonius and Vorberg (1994), Dzhafarov (1992), Fisher and Goldstein (1983), Goldstein and Fisher (1991, 1992), Kounios (1993), Meyer, Yantis, Osman, and Smith, (1984) Miller (1982), Shaked and

Shanthikumar (1994), Sternberg (1973); Stoyan (1983), Townsend (1990b), and Vorberg (1981). For a survey of results in scheduling theory, see Righter (1994).

Our interest here is in using cumulative distribution functions to recover the process architecture under the assumption of selective influence. Two different approaches to this problem have been pursued. One technique, developed by Ashby and Townsend (1980) and Roberts and Sternberg (1992), is a test of whether two factors selectively influence processes whose durations are combined by addition to form the response time. For a critical analysis, see Van Zandt and Ratcliff (1995). A generalization of this technique is presented in Dzhafarov and Schweickert (1995), Dzhafarov and Cortese (1996), and Cortese and Dzhafarov (1996). With the generalization one can test not only for the operation of addition, but for virtually any operation that is commutative and associative.

The other technique was developed by Nozawa (1992) and Townsend and Nozawa (1995). With this technique, interactions of cumulative distribution functions are examined for evidence that factors selectively influence processes that are (a) in series or (b) in parallel; the influenced processes are followed by an AND gate or an OR gate, and perhaps followed by an additional process. Here we generalize this work to serial-parallel networks, a type of directed acyclic network. For comparison, we begin with an example of selective influence in a simple architecture that is not a directed acyclic network. Then we describe directed acyclic networks, and then we describe the technique applied to them.

A mixture model. Roberts and Sternberg (1992) described a simple mixture model for information processing. Suppose a task is carried out using process a with probability p and using process b with probability $(1-p)$. Let the duration of a have cumulative distribution function $F_a(t)$, and let the duration of b have cumulative distribution function $F_b(t)$. Then the cumulative distribution for the response time is $G(t) = pF_a(t) + (1-p)\,F_b(t)$.

Now suppose one experimental factor selectively influences process a, so when this factor is at level i, $i = 1, 2$, the cumulative distribution function for process a is $F_{ai}(t)$. The cumulative distribution function for process b does not change when levels of this factor change. Likewise, suppose another factor selectively influences process b. When this factor is at level j, $j = 1, 2$, the cumulative distribution function for process b is $F_{bj}(t)$. The cumulative distribution function for process a does not change when levels of this factor change. When the first factor is at level i and the second is at level j, the cumulative distribution for the response time is

$$G_{ij}(t) = pF_{ai}(t) + (1-p)\,F_{bj}(t).$$

For any time t, define an interaction contrast

$$c(t) = G_{22}(t) - G_{21}(t) - G_{12}(t) + G_{11}(t).$$

It is easy to see that for every t, $c(t) = 0$. This provides a simple test for the hypothesis that the factors selectively influence processes arranged in a mixture model.

We turn now to a class of models making different predictions about the interaction contrasts. Eventually, one may be able to compare many models on the basis of their predictions about selective influence and cumulative distribution functions. At this time, it remains to be seen whether other models make other distinctive predictions, and the number of models that can be compared in this way is limited. However, within the class of directed acyclic network models, the technique can be used to distinguish some process arrangements from others, and that is the use we describe here.

DIRECTED ACYCLIC NETWORKS

Suppose a subject performing a task executes a set of mental processes arranged in a network as in Fig. 1. Each process is represented by a node and has a non-negative duration which varies from trial to trial. If one process immediately precedes another, an arrow is drawn from the node representing the former process to the node representing the latter process. The task begins at a node, o, called the *source*, which indicates the presentation of the stimulus. No node precedes the source. Likewise, the task ends at a node, r, called the *terminus*, which indicates the onset of the response. No node follows the terminus.

A process a *precedes* another process b if there is a path starting at the node representing a, proceeding along some arc in the direction indicated by the arrow to another node, and then perhaps proceeding along further arcs and nodes, always in the directions indicated by the arrows, and ending at the node representing b. We say processes a and b are *sequential* if either a precedes b or b precedes a. Two processes a and b are *concurrent* if they are not sequential.

If a process can begin execution when any one of its immediate predecessors has completed, the process is released with an OR gate. If a process can begin execution only when all of its immediate predecessors have completed, the process is released with an AND gate. We consider two types of networks here: OR networks, in which every gate is an OR gate and AND networks, in which every gate is an AND gate. The response is made at the terminal node, and we assume the terminal node is an OR gate or an AND gate, of the same type as the others. Networks having both OR and AND gates, or with other types of gates, are possible, of course, but are beyond the scope of this paper.

We assume the clock starts when the stimulus is presented at the source node. The response time is the time at which the response is made at the terminal node. If all the gates are OR gates, the response time will be the sum of the durations of all the processes on the shortest path (in terms of total duration) from the source to the terminus. If all the gates are AND gates, the response time will be the sum of the durations of all the processes on the longest path (in terms of total duration) through the network. The longest path through the network is called the *critical path*, and a directed acyclic network with AND gates is sometimes called a *critical path network* or a *PERT network*.

In a network with AND gates or OR gates, a path directed from a node to itself would indicate that the process represented by the node must finish (AND gates) or could finish (OR gates) before it could start. This is impossible, so we assume no process precedes itself, immediately or otherwise. We assume, in short, that the task network is a directed acyclic network. A more extensive introduction can be found in Fisher and Glasser (1996) or Schweickert, Fisher and Goldstein (1992). An introduction to several other network models of processing is given in Liu (1996).

Serial-Parallel Networks

This paper is primarily about serial-parallel networks. The network in Fig. 1 is an example. A directed acyclic network is serial-parallel if it can be constructed with the following procedure. We begin with a single node. A single node is the simplest possible serial-parallel network, and we may stop the procedure at this point. Otherwise, we may combine two serial-parallel networks N_1 and N_2 in one of the following ways. (a) We may combine them in parallel, that is, so that no node in N_1 precedes any node in N_2 and no node in N_2 precedes any node in N_1. (b) We may combine them in series, that is, so that every node in N_1 precedes every node in N_2 or else every node in N_2 precedes every node in N_1. We may continue combining serial-parallel networks, carrying out either step (a) or step (b), and stop after a finite number of steps. For more background, see Martin (1965) and Mohring (1982).

The serial-parallel network illustrated in Fig. 1 can readily be seen to have been constructed in this way. The procedure began with the single node x. Then y was placed in series with x. Then z was placed in parallel with the network formed by x and y. Let N_1 denote the network constructed so far, and let N_2 denote a new network consisting of the single node r. Networks N_1 and N_2 were placed in series; that is, arcs were drawn so that z precedes r and y precedes r (x also precedes r, then, because y precedes r). The last step was to put the network constructed so far in series (and following) a network consisting of the single node o. In the completed serial-parallel network, o is the source node and r is the terminal node.

Directed acyclic networks that are not serial-parallel networks are easily characterized. The network in Fig. 2 is called a Wheatstone bridge. Kaerkes and Mohring (1978) and Dodin (1985) have shown that a directed acyclic network is a serial-parallel network if and only if it has no subnetwork in the form of a Wheatstone bridge. An attempt to construct a Wheatstone bridge with the serial-parallel procedure will show that it cannot be done. The more interesting fact is that if the serial-parallel procedure cannot be used to construct a certain network, then that network must contain within it a subnetwork in the form of a Wheatstone bridge.

Many interesting information processing models are not serial-parallel networks. For example, models of dual tasks that postulate two or more bottlenecks very often employ a Wheatstone bridge and hence are not serial-parallel networks; see, e.g., De Jong (1993), Ehrenstein, Schweickert, Choi, and Proctor (1997), and Schweickert (1978). Cumulative distribution functions for the Wheatstone bridge are beyond the scope of this paper, but are discussed in Schweickert and Giorgini (1999).

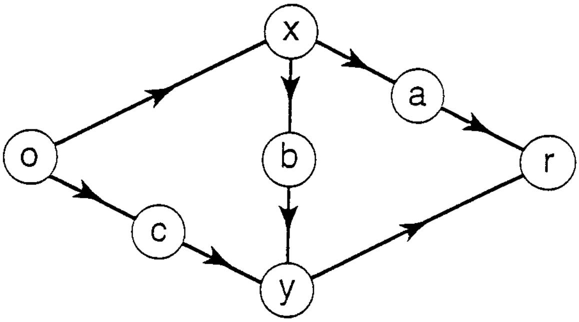

FIG. 2. A Wheatstone bridge. A directed acyclic network is erial-parallel if and only if it does not contain a subnetwork in the form of a Wheatsone bridge.

Stochastic independence. Consider the set of random variables $Y_1, Y_2, ..., Y_n$. If the joint distribution of every subset S of the random variables is the product of the marginal distributions of the random variables in S, then $Y_1, Y_2, ..., Y_n$ are called *mutually independent.* For convenience, we will call a network in which the process durations are mutually independent an *independent network.*

Notation and assumptions about distributions. We denote the duration of a process z in the network as T_z, its cumulative distribution function (cdf) as $F_z(t) = P[T_z \leqslant t]$, and the corresponding probability density function as $f_z(t)$. The survivor function is $\overline{F_z}(t) = 1 - F_z(t)$.

For all random variables, a density function is assumed to exist and to have at most a finite number of points of infinite discontinuity. A point t_0 is a point of infinite discontinuity of $f(t)$ if (a) as t approaches t_0 from the left the limit of $f(t)$ is $+\infty$ or $-\infty$, (b) as t approaches t_0 from the right the limit of $f(t)$ is $+\infty$ or $-\infty$, or (c) $f(t)$ oscillates near t_0 and there is no neighborhood of t_0 in which $f(t)$ is bounded (Carslaw, 1930). All distributions are assumed to be continuous, that is, to have no points of positive mass (Feller, 1971). Each random variable is assumed to have a finite mean.

We denote the cumulative distribution function for the overall response time RT as $G(t)$ and the corresponding density as $g(t)$.

Writing the distribution functions. It is easy to write a formula for the cdf, $P[RT \leqslant t]$, of an independent series-parallel network. For two processes a and b in parallel, followed by a response made at an AND gate,

$$\begin{aligned} G_{\text{AND}}(t) &= P[RT \leqslant t] = P[T_a \leqslant t \text{ AND } T_b \leqslant t] \\ &= P[T_a \leqslant t]\, P[T_b \leqslant t] \\ &= F_a(t)\, F_b(t). \end{aligned}$$

By a similar calculation, $\bar{G}_{\text{OR}}(t) = \bar{F}_a(t)\, \bar{F}_b(t)$. Now consider a network consisting of two processes a and b in series. The cdf of $RT = T_a + T_b$ is calculated as follows (see, e.g., Feller, 1971; Luce, 1986; Townsend & Ashby, 1983). (Because the two processes are in series, it does not matter whether we consider the network to have AND gates or OR gates.)

$$\begin{aligned} G_{\mathrm{AND}}(t) = G_{\mathrm{OR}}(t) &= P[T_a + T_b \leqslant t] \\ &= \int_0^\infty P[T_a \leqslant x \;\&\; T_b = t - x]\, dx \\ &= \int_0^\infty F_a(x)\, f_b(t-x)\, dx. \end{aligned} \tag{1}$$

Feller (1971, p. 145) calls the expression above the *convolution* of $F_a(t)$ and $F_b(t)$ and denotes it $F_a(t) \star F_b(t)$. The term convolution is also used for a similar, but different expression. The density function for $T_a + T_b$ is

$$g(t) = \int_0^\infty f_a(x)\, f_b(t-x)\, dx.$$

This expression is called the *convolution* of $f_a(t)$ and $f_b(t)$ by Feller and others and denoted $f_a(t)^\star f_b(t)$.

Consider, as an example, a network consisting of two parallel processes, a and b, followed by a third process c which starts at an AND gate. One begins by forming the cdf for a and b as a product $F_a(t) \cdot F_b(t)$. The convolution of this cdf with the cdf for process c forms the cdf for the complete network,

$$[F_a(t) \cdot F_b(t)] \star F_c(t).$$

If the network was larger, one would continue by taking products for components in parallel and convolutions for components in series.

As another example, consider the network in Fig. 1. To keep the example simple, suppose the durations of processes o and r are fixed at 0. For the subnetwork formed by placing processes x and y in series, the cdf is $F_x(t) \star F_y(t)$. Process z is in parallel with this subnetwork, making the cdf for the completion time of the entire network

$$\mathrm{G}_{\mathrm{AND}}(t) = [F_x(t) \star F_y(t)] \cdot F_z(t), \tag{2}$$

when the network has an AND gate. When the network has an OR gate, the survivor function for the completion time of the entire network is

$$\bar{\mathrm{G}}_{\mathrm{OR}}(t) = [\overline{F_x(t) \star F_y(t)}] \cdot \bar{F}_z(t).$$

After rearranging, the corresponding cdf is seen to be

$$G_{\mathrm{OR}}(t) = F_x(t) \star F_y(t) + F_z(t) - [F_x(t) \star F_y(t)]\, F_z(t). \tag{3}$$

EFFECTS OF SELECTIVELY INFLUENCING A SINGLE PROCESS

Consider an experimental factor which selectively influences a process a in a task network. We denote the duration of process a when the factor is at level i as T_{ai}, for $i = 1, 2, 3, \ldots$, with the levels numbered so that the process becomes more difficult to carry out as the level numbers increase. For convenience we will consider only two levels, although our results apply for more than two.

We usually assume the mean duration of the process increases as the factor levels increase, but our topic is the response time distributions, and the assumption that the means are ordered implies little about the distributions as a whole (e.g., Townsend, 1990a). Here we make the stronger assumption that the factor produces stochastic dominance for the process durations. That is, if $F_{ai}(t) = P[T_{ai} \leqslant t]$ is the cdf of T_{ai}, then we assume for every t, $F_{a2}(t) \leqslant F_{a1}(t)$. We say T_{a2} is stochastically larger than T_{a1}, $T_{a2} \geqslant_{st} T_{a1}$, and it follows that the expected value of T_{a2} is greater than or equal to that of T_{a1}. A condition equivalent to stochastic dominance is presented in Townsend and Schweickert (1989).

By saying the factor selectively influences process a we mean here that the factor (a) changes the marginal distributions of process a, (b) changes the marginal distribution of no process other than a (*marginal independence*), and (c) preserves the mutual independence of the process durations, that is, the process durations are mutually independent at all levels of the factor.

More notation. When a factor selectively influencing process a is at level i, we denote the density function of the duration of a as $f_{ai}(t)$. We denote the density function of the response time as $g_i(t)$, and the corresponding cdf as $G_i(t)$. If $N^\star$ is a subnetwork of a network N, one can imagine the nodes in $N^\star$ as preceeded by a starting node of zero duration and followed by a terminal node of zero duration. One can then consider the completion time for $N^\star$ by itself; it has density $g^\star(t)$ and cdf $G^\star(t)$. For example, one can consider the subnetwork in Fig. 1 consisting of processes x and y. The completion time for this subnetwork is the duration of x plus the duration of y.

Effects on cdfs. It may seem at first that if a process involved in a task is prolonged, the reaction time for the task must be prolonged as well. Empirically, this does not always happen. For example, in a dual task, increasing the interval between the onset of the first stimulus and the onset of the second sometimes decreases the response time to the first stimulus (Lien, Allen, & Smith, 1998; Van Selst & Jolicoeur, 1994, Experiment 2). A plausible explanation is that full capacity can be allocated to the first stimulus until the second stimulus is presented. A delay in the onset of the second stimulus allows full capacity processing of the first stimulus to continue longer, thus speeding the first response.

Of course, the interval between two stimuli is not a typical process. A different example of how prolonging a process could lead to a decrease in reaction time is illustrated in Fig. 3. Processes are represented as rectangles; the length of a rectangle indicates the duration of the corresponding process. The system has two processors and uses the scheduling rule that a processor cannot be idle if a process is available for it. Suppose for the task illustrated that processes a and d must be executed on processor 1 (first row of rectangles). Process c cannot begin until either a or b is finished. Process d cannot begin until both a and b are finished. In the top panel, when a is short, process c occupies processor 1, delaying access by process d. When a is longer, process c occupies processor 2, leading to a faster reaction time (bottom panel). The upshot is that an increase in the duration of process a leads to a decrease in reaction time. The example shows that simple scheduling rules sometimes lead to counterintuitive outcomes; other examples are discussed by Graham (1978).

As a final example of the need for care in extrapolating from the stochastic dominance of random variables to stochastic dominance of functions taking them as arguments, note that it is possible that $X_1 \leqslant_{st} Y_1$ and $X_2 \leqslant_{st} Y_2$ for random variables X_1, X_2, X_1, and Y_2, while it is false that $X_1 + X_2 \leqslant_{st} Y_1 + Y_2$ (Ross, 1983, p. 279, Exercise 8.2). The failure can occur when the random variables are correlated.

The following theorem shows that processes in independent serial-parallel AND and OR networks are well behaved in the sense that if a factor produces stochastic dominance for the distributions of the duration of a single process, then the factor produces stochastic dominance for the distributions of the response times. The proof introduces the inductive procedure we will use throughout. Because of this theorem, if $F_{a1}(t) \geqslant F_{a2}(t)$ for all t when a factor selectively influences process a, we simply say that the factor produces stochastic dominance.

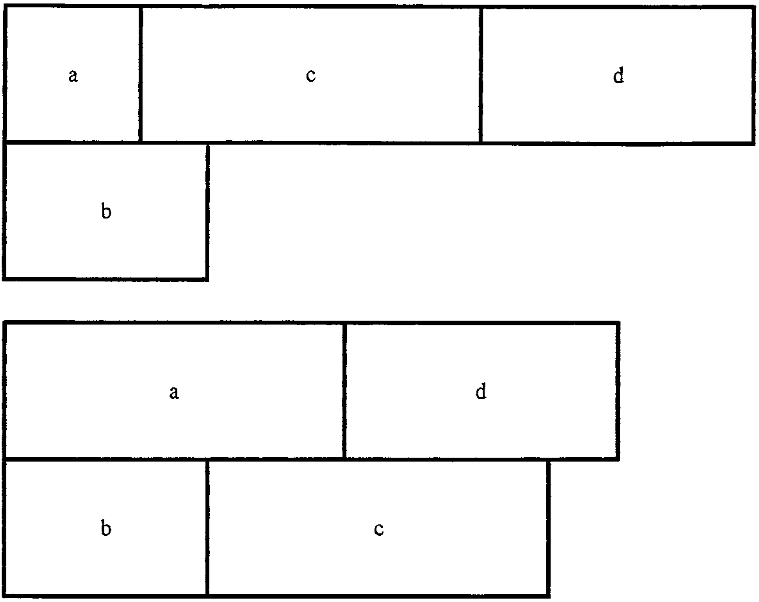

FIG. 3. A system in which a processor is not allowed to be idle if a process is available for it. In each panel, rectangles in the first row represent processes scheduled on processor 1; those in the second row represent processes scheduled on processor 2. In the top panel, process a is relatively short, but the completion time of the task is long.

THEOREM 1. *Suppose a factor selectively influences a process a in an independent serial-parallel network, in which all the gates are AND gates or all the gates are OR gates. Suppose for all* $t < \tau$, $F_{a1}(t) \geqslant F_{a2}(t)$. *Then for all* $t \leqslant \tau$, $G_1(t) \geqslant G_2(t)$.

Proof. The proof is by induction on the number of processes in the network. If a is the only process, clearly the conclusion is true.

Suppose the conclusion is true for independent serial-parallel networks with n or fewer processes. Consider an independent serial-parallel network N with $n+1$ processes. There are two cases.

Case 1. Suppose network N consists of a subnetwork $N_a^\star$ having n or fewer processes, one of which is a, and a subnetwork $N^{\star\star}$ in parallel with $N_a^\star$. Let the cdf of the completion time of the subnetwork $N_a^\star$ be $G_i^\star(t)$ when the factor is at level i. Let the cdf of the completion time of the subnetwork $N^{\star\star}$ be $G^{\star\star}(t)$.

If there are AND gates,

$$\begin{aligned} G_i(t) &= G_i^\star(t)\, G^{\star\star}(t), \qquad \text{so} \\ G_1(t) - G_2(t) &= [G_1^\star(t) - G_2^\star(t)]\, G^{\star\star}(t) \\ &\geqslant 0, \end{aligned}$$

for all $t < \tau$.

If there are OR gates,

$$G_i(t) = G_i^\star(t) + G^{\star\star}(t) - G_i^\star(t)\, G^{\star\star}(t).$$

Then

$$\begin{aligned} G_1(t) - G_2(t) &= G_1^\star(t) - G_2^\star(t) - [G_1^\star(t) - G_2^\star(t)]\, G^{\star\star}(t) \\ &= [G_1^\star(t) - G_2^\star(t)][1 - G^{\star\star}(t)] \\ &\geqslant 0, \end{aligned}$$

for all $t \leqslant \tau$.

Case 2. Suppose network N consists of a subnetwork $N_a^\star$ with n or fewer processes, one of which is a, and a subnetwork $N^{\star\star}$ in series with $N_a^\star$. Let the density of the completion time of the subnetwork $N^{\star\star}$ be $g^{\star\star}(t)$.

Whether there are AND gates or OR gates,

$$G_i(t) = \int_0^t G_i^\star(x)\, g^{\star\star}(t-x)\, dx.$$

By the induction hypothesis, $G_1^\star(t) \geqslant G_2^\star(t)$.

Then

$$G_1(t) - G_2(t) = \int_0^t [G_1^\star(x) - G_2^\star(x)]\, g^{\star\star}(t-x)\, dx$$

$$\geqslant 0,$$

for all $t \leqslant \tau$. Q.E.D.

One immediate consequence of the theorem is that there is no time τ' such that $g_1(t) < g_2(t)$ for all $t \leqslant \tau'$. Another is the following.

COROLLARY. *In both independent AND networks and independent OR networks, if for all $t < \tau$, $f_{a1}(t) \geqslant f_{a2}(t)$, then for all $t < \tau$, $G_1(t) \geqslant G_2(t)$.*

Effects on density functions. According to the preceding theorem, the assumption that a factor produces stochastic dominance for the duration of a process implies that the factor produces stochastic dominance for the duration of a network containing that process. Unfortunately, assumptions about the density function for a process do not have strong implications about the density function for a network containing the process. The following theorem shows that for times between 0 and some critical value in AND networks, a constraint on the density functions for a process leads to a similar constraint on the density functions for a network containing the process. The theorem is stated for AND networks. In the Appendix we briefly discuss the difficulty in applying it to OR networks.

THEOREM 2. *Suppose process a in an independent serial-parallel AND network is selectively influenced by a factor. If there is a time τ such that for all $t < \tau$*

$$f_{a1}(t) \geqslant f_{a2}(t),$$

then for all $t < \tau$

$$g_1(t) \geqslant g_2(t).$$

Proof. If a is the only process in the network, the proof is finished.

Otherwise, suppose the statement is true for independent serial-parallel AND networks with n or fewer processes. Consider an independent serial-parallel AND network N with $n+1$ processes.

Case 1. Suppose N consists of a subnetwork $N_a^\star$ with n or fewer processes, one of which is a, and a subnetwork $N^{\star\star}$ in parallel with $N_a^\star$.

Let $G_i^\star(t)$ be the cdf of the completion time of $N_a^\star$ when the factor is at level i, $i = 1, 2$. Let $G^{\star\star}(t)$ be the cdf of the completion time of $N^{\star\star}$, and let $g^{\star\star}(t)$ be the corresponding density. For $i = 1, 2$,

$$G_i(t) = G_i^\star(t)\, G^{\star\star}(t),$$

so then

$$g_i(t) = g_i^\star(t)\, G^{\star\star}(t) + G_i^\star(t)\, g^{\star\star}(t).$$

We assumed $g_1^\star(t) \geqslant g_2^\star(t)$ for $t \leqslant \tau$, by the induction hypothesis, so $G_1^\star(t) \geqslant G_2^\star(t)$ for $t \leqslant \tau$.
Then

$$\begin{aligned} g_1(t) - g_2(t) &= g_1^\star(t)\, G^{\star\star}(t) + G_1^\star(t)\, g^{\star\star}(t) \\ &\quad - g_2^\star(t)\, G^{\star\star}(t) - G_2^\star(t)\, g^{\star\star}(t) \\ &= [\, g_1^\star(t) - g_2^\star(t)]\, G^{\star\star}(t) + [\, G_1^\star(t) - G_2^\star(t)]\, g^{\star\star}(t). \end{aligned}$$

Then $g_1(t) - g_2(t) \geqslant 0$ for $t < \tau$.

Case 2. Suppose N consists of a subnetwork $N_a^\star$ having n or fewer processes, one of which is a, and a subnetwork $N^{\star\star}$ in series with $N_a^\star$.

The notation is as in Case 1.

Then

$$g_i(t) = \int_0^t g_i^\star(x)\, g^{\star\star}(t-x)\, dx$$

so

$$g_1(t) - g_2(t) = \int_0^t (g_1^\star(x) - g_2^\star(x))\, g^{\star\star}(t-x)\, dx.$$

We assume in the induction hypothesis that $g_1^\star(x) - g_2^\star(x) \geqslant 0$ for $x \leqslant \tau$. Then $g_1(t) - g_2(t) \geqslant 0$ for $t \leqslant \tau$. Q.E.D.

Remark. Note that, if the difference $g_1(t) - g_2(t)$ starts positive, it changes sign later. For suppose $g_1(t) > g_2(t)$ over some interval. If $g_1(t) - g_2(t) > 0$ for all t, then

$$1 = \int_0^\infty g_1(t)\, dt > \int_0^\infty g_2(t)\, dt = 1$$

which is impossible. It follows that for some t, $g_1(t) - g_2(t) < 0$.

EFFECTS OF SELECTIVELY INFLUENCING TWO PROCESSES

When two processes are selectively influenced, the resulting cumulative distribution functions behave differently depending on whether the processes are concurrent or sequential and whether the gates are AND gates or OR gates. Suppose a and b are two processes in a series-parallel network. When factor 1 is at level i, let the duration of a be T_{ai}, and when factor 2 is at level j, let the duration of b be T_{bj}.

Let $F_{ai}(t)$ be the cdf of T_{ai} and let $F_{bj}(t)$ be the cdf of T_{bj}. In our definition of selective influence, we assume $F_{ai}(t)$ is invariant over all levels j of factor 2 and $F_{bj}(t)$ is invariant over all levels i of factor 1. Let $G_{ij}(t)$ be the task completion cdf in condition (i, j), that is, the condition in which factor 1 is at level i and factor 2 is at level j. The topic of this section is the cumulative distribution function interaction contrast,

$$c(t) = G_{22}(t) - G_{21}(t) - G_{12}(t) + G_{11}(t).$$

Nozawa (1992) and Townsend and Nozawa (1995) pointed out that if the task network consists of two processes in series, then the net area bounded by the cdf interaction contrast and the time axis will be 0. To see this, let $E[RT_{ij}]$ be the expected value of the response time when factor 1 is at level i and factor 2 is at level j. Because the response times are nonnegative,

$$E[RT_{ij}] = \int_0^\infty [1 - G_{ij}(t)]\, dt \tag{4}$$

(e.g., Cinlar, 1975). From Eq. (4), the interaction contrast for means can be written

$$E[RT_{22}] - E[RT_{21}] - E[RT_{12}] + E[RT_{11}] = -\int_0^\infty c(t)\, dt.$$

For the two processes in series we are considering,

$$E[RT_{22}] - E[RT_{21}] - E[RT_{12}] + E[RT_{11}] = 0.$$

It follows that

$$\int_0^\infty c(t)\, dt = 0.$$

Nozawa (1992) and Townsend and Nozawa (1995) also showed that when the two processes selectively influenced are in parallel, possibly followed by another process, the cdf interaction contrast will be positive for an AND gate and negative for an OR gate. This result of Nozawa and Townsend for parallel processes is an essential component of our results for concurrent processes in serial-parallel networks. Because we assume independence throughout, we only need the version of the theorem by Townsend and Nozawa (1995) that makes this assumption. Townsend and Nozawa (1995) also provide a proof for interdependent processes under additional constraints.

Concurrent Processes

THEOREM 3 (Nozawa & Townsend). *Suppose an independent serial-parallel network consists of two processes a and b in parallel. Suppose one factor selectively* *influences a and another selectively influences b. Suppose for every t,* $F_{a1}(t) \geqslant F_{a2}(t)$, $F_{b1}(t) \geqslant F_{b2}(t)$, *and there is an interval I where both inequalities are strict. With an AND gate, for every t,*

$$G_{22}(t) - G_{21}(t) - G_{12}(t) + G_{11}(t) \geqslant 0,$$

and over the interval I the inequality is strict.

With an OR gate, the conclusion holds with the inequality reversed; that is,

$$G_{22}(t) - G_{21}(t) - G_{12}(t) + G_{11}(t) \leqslant 0.$$

Proof. Suppose there is an AND gate. Then

$$\begin{aligned} G_{ij}(t) &= P[T_{ai} \leqslant t \text{ AND } T_{bj} \leqslant t] = P[T_{ai} \leqslant t]\, P[T_{bj} \leqslant t] \\ &= F_{ai}(t)\, F_{bj}(t). \end{aligned}$$

Then

$$\begin{aligned} c(t) &= G_{22}(t) - G_{21}(t) - G_{12}(t) + G_{11}(t) \\ &= (F_{a2}(t) - F_{a1}(t))(F_{b2}(t) - F_{b1}(t)) \\ &\geqslant 0. \end{aligned}$$

Clearly, when $t \in \mathrm{I}$, the inequality is strict.

Suppose the gate is an OR gate. Then

$$G_{ij}(t) = P[T_{ai} < t \text{ OR } T_{bj} < t] = F_{ai}(t) + F_{bj}(t) - F_{ai}(t)\, F_{bi}(t),$$

and the conclusion $c(t) < 0$ follows. Q.E.D.

The next theorem generalizes this result to concurrent processes in an arbitrary serial-parallel network. Before presenting it, we illustrate it for the network in Fig. 1, considered as an AND network in Fig. 4, and considered as an OR network in Fig. 5. We assume each process has an exponential distribution, that is, the density function for the duration of a process w is $f_w(t) = \lambda e^{-\lambda t}$, and the cumulative distribution function is $F_w(t) = P[T_w < t] = 1 - e^{-\lambda t}$, where $1/\lambda$ is the mean duration of process w. The mean durations for processes u, y, and z are given in the figure captions. Process durations producing pronounced effects for an AND gate tend to produce small effects for an OR gate, and vice versa, so to illustrate the results clearly, different process durations are used in different examples.

The cumulative distribution functions were calculated as follows for the network in Fig. 1. Let the mean durations of processes x, y, and z be $1/\alpha$, $1/\beta$, and $1/\gamma$, respectively. For the subnetwork consisting of x and y in series, Eq. 1 leads to

$$F_x(t) \star F_y(t) = [\alpha/(\alpha + \beta)](1 - e^{-\beta t}) + [\beta/(\alpha + \beta)](1 - e^{-\alpha t})$$

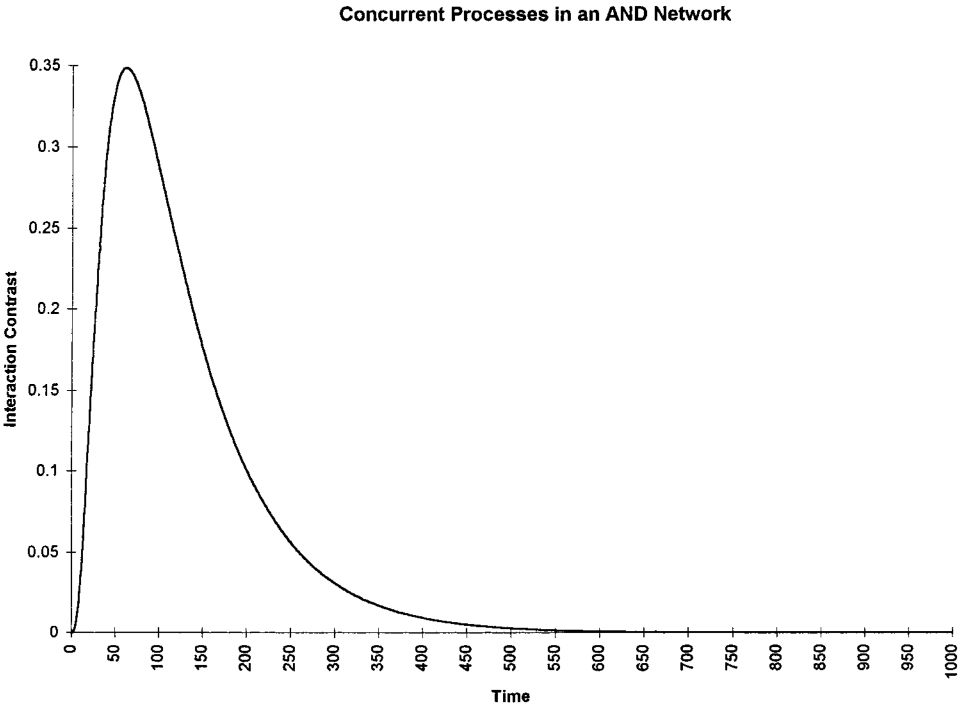

FIG. 4. When the network in Fig. 1 has an AND gate, the interaction contrast function for concurrent processes x and z is positive. Processes o and r have fixed duration 0. All other processes have exponential durations. Process x has means 10 and 100, process z has means 50 and 500. Process y has mean 12.5.

(also see, e.g., Fisher & Goldstein, 1983, p. 144). When the network in Fig. 1 is considered to have an AND gate, the cumulative distribution function for the response time is given by Equation 2, which leads to

$$G_{\text{AND}}(t) = [[\alpha/(\alpha+\beta)](1-e^{-\beta t}) + [\beta/(\alpha+\beta)](1-e^{-\alpha t})](1-e^{-\gamma t}).$$

When the network in Fig. 1 is considered to have an OR gate, the cumulative distribution function for the response time can be calculated from Eq. (3), in a similar manner. This ends the example and we turn now to Theorem 4.

The proofs in the remainder of the paper are based on a classification of the possible networks into cases. Under the assumptions of some theorems, certain cases are impossible. They are mentioned nonetheless for the sake of completeness and to facilitate comparison.

THEOREM 4. *Suppose processes a and b are concurrent in an independent serial-parallel network. Suppose one factor selectively influences a and another selectively influences b. Suppose for every t,* $F_{a1}(t) \geqslant F_{a2}(t)$ *and* $F_{b1}(t) \geqslant F_{b2}(t)$.

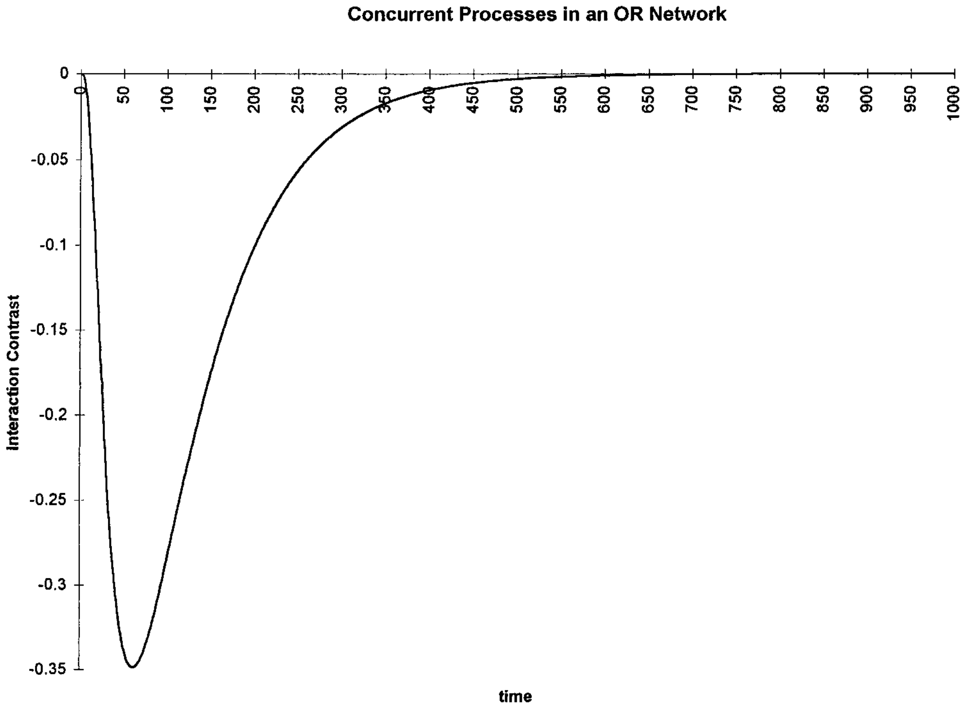

FIG. 5. When the network in Fig. 1 has an OR gate, the interaction contrast function for the concurrent processes x and z is negative. Process means and distributions are as in Fig. 4.

If all the gates are AND gates, then for every t,

$$G_{22}(t) - G_{21}(t) - G_{12}(t) + G_{11}(t) \geqslant 0.$$

If all the gates are OR gates, the inequality in the conclusion is reversed.

Proof. If a and b are the only processes in the network, Theorem 3 applies.

Suppose the proposition is true when the number of processes in the network is n or fewer. Consider an independent serial-parallel network N with $n+1$ processes. Suppose all the gates are AND gates. It is convenient to consider four cases.

Case 1. Suppose N consists of a subnetwork $N^{\star}_{ab}$ with n processes or fewer, containing both a and b, and another subnetwork $N^{\star\star}$ in parallel with the entire subnetwork $N^{\star}_{ab}$. Let $t^{\star}_{ij}$ be the duration of $N^{\star}_{ab}$, when the factor influencing process a is at level i and the factor influencing b is at level j. Let $t^{\star\star}$ be the completion time of the subnetwork $N^{\star\star}$.

$$G_{ij}(t) = P[T^{\star}_{ij} \leqslant t \text{ AND } T^{\star\star} \leqslant t] = P[T^{\star}_{ij} \leqslant t]\; P[T^{\star\star} \leqslant t]$$

Then

$$c(t) = [P[T^{\star}_{22} \leqslant t] - P[T^{\star}_{21} \leqslant t] - P[T^{\star}_{12} \leqslant t] + P[T^{\star}_{11} \leqslant t]]\, P[T^{\star\star} \leqslant t]$$
$$\geqslant 0$$

by the induction hypothesis.

Case 2. Suppose the situation is as in Case 1, but subnetwork $N^{\star\star}$ is in series with the subnetwork $N^{\star}_{ab}$.

$$G_{ij}(t) = P[T^{\star}_{ij} + T^{\star\star} \leqslant t] = \int_0^t g^{\star\star}(x)\, G^{\star}_{ij}(t-x)\, dx$$

Then

$$c(t) = \int_0^t g^{\star\star}(x)[G^{\star}_{22}(t-x) - G^{\star}_{21}(t-x) - G^{\star}_{12}(t-x) + G^{\star}_{11}(t-x)]\, dx.$$

Since $g^{\star\star}(x)$ is a density function, $g^{\star\star}(x) \geqslant 0$. By the induction hypothesis,

$$G^{\star}_{22}(t) - G^{\star}_{21}(t) - G^{\star}_{12}(t) + G^{\star}_{11}(t) \geqslant 0.$$

Then $c(t) \geqslant 0$.

If neither Case 1 nor Case 2 applies, N does not consist of a subnetwork $N^{\star\star}$ in series or in parallel with a subnetwork having n or fewer processes, a and b among them.

Case 3. Network N consists of a subnetwork $N^{\star}_a$ containing a in parallel with a subnetwork $N^{\star\star}_b$ containing b. Let $T^{\star}_i$ be the duration of the subnetwork $N^{\star}_a$ when factor 1 is at level i; $T^{\star\star}_j$ is defined similarly. Then

$$G_{ij}(t) = P[T^{\star}_i \leqslant t \text{ AND } T^{\star\star}_j \leqslant t]$$
$$= G^{\star}_i(t)\, G^{\star\star}_j(t).$$

By Theorem 1, $G^{\star}_1(t) \geqslant G^{\star}_2(t)$, and $G^{\star\star}_1(t) \geqslant G^{\star\star}_2(t)$. The conclusion follows.

Case 4. Network N consists of a subnetwork containing a in series with a subnetwork containing b. This is not possible, because a and b are concurrent.

The proof is similar if all the gates are OR gates. Q.E.D.

Sequential Processes

The behavior of the cdf interaction contrast is quite different when sequential processes, rather than concurrent processes, are selectively influenced. To focus on the key differences, suppose for a moment there are no intervals over which the cdf interaction contrast is zero. The cdf interaction contrast for concurrent processes is relatively simple. It has the same sign at all times, positive for AND networks and negative for OR networks. For sequential processes, we will see that for AND networks, the interaction contrast is positive for a short interval of time after 0, and then it changes sign. Moreover, the area bounded by the cdf interaction contrast and the time axis beyond an arbitrary time $t>0$ is less than or equal to zero. For OR networks, the cdf interaction contrast cannot be negative over an interval close to time 0. For any time t greater than 0, the area from 0 to t bounded by the cdf interaction contrast and the time axis is greater than or equal to 0.

Times near zero. If one of the density functions $f_{a1}(t)$ and $f_{a2}(t)$ lies above the other at some point, and then at a later point the one previously lying above now lies below, we will say the densities have *completely crossed.* The following theorem is based on the assumption that there is a time $\tau>0$ such that $f_{a1}(t)$ and $f_{a2}(t)$ do not completely cross in the interval $(0, \tau)$. Note that because $F_{a1}(t) \geqslant F_{a2}(t)$ for all t, if there is such a τ it must be that $f_{a1}(t) \geqslant f_{a2}(t)$ for all t in $(0, \tau)$.

THEOREM 5. *Suppose a and b are sequential processes in an independent serial-parallel AND network. Suppose one factor selectively influences a and another selectively influences b. Suppose for all t, $F_{a1}(t) \geqslant F_{a2}(t)$ and $F_{b1}(t) \geqslant F_{b2}(t)$. Suppose there is a time $\tau>0$ such that $f_{a1}(t)$ and $f_{a2}(t)$ do not completely cross in the interval* $(0, \tau)$.

Then for all t, $0<t<\tau$, $G_{22}(t)-G_{21}(t)-G_{12}(t)+G_{11}(t) \geqslant 0$.

Proof. Suppose a and b are the only two processes. Then

$$\begin{aligned} c(t) &= \int_0^t [f_{a2}(x)-f_{a1}(x)][F_{b2}(t-x)-F_{b1}(t-x)]\,dx \\ &\geqslant 0 \end{aligned}$$

for $t<\tau$.

Suppose the statement is true for independent serial-parallel networks with n processes or fewer. Consider an independent serial-parallel network N with $n+1$ processes.

Case 1. Network N consists of a subnetwork $N^{\star}_{ab}$ having n or fewer processes, a and b among them, and another subnetwork $N^{\star\star}$ in parallel with $N^{\star}_{ab}$.

We use the same notation as before. In particular, $G^{\star}_{ij}(t)$ is the cdf of the completion time of the subnetwork $N^{\star}_{ab}$ and $c^{\star}(t)=G^{\star}_{22}(t)-G^{\star}_{21}(t)-G^{\star}_{12}(t)+G^{\star}_{11}(t)$. Then

$$\begin{aligned} G_{ij}(t) &= G^{\star}_{ij}(t)\, G^{\star\star}(t). \\ c(t) &= [G^{\star}_{22}(t)-G^{\star}_{21}(t)-G^{\star}_{12}(t)+G^{\star}_{11}(t)]\, G^{\star\star}(t) \\ &= c^{\star}(t)\, G^{\star\star}(t) \\ &\geqslant 0 \end{aligned}$$

for $0<t<\tau$.

Case 2. Network N consists of a subnetwork $N^{\star}_{ab}$ having n or fewer processes, a and b among them, and another subnetwork $N^{\star\star}$ in series with $N^{\star}_{ab}$. Then

$$G_{ij}(t) = \int_0^t G^{\star}_{ij}(x)\, g^{\star\star}(t-x)\, dx.$$

Then

$$\begin{aligned} c(t) &= \int_0^t c^{\star}(x)\, g^{\star\star}(t-x)\, dx. \\ &\geqslant 0 \end{aligned}$$

for $0 < t < \tau$.

Case 3. Network N consists of a subnetwork containing process a in parallel with a subnetwork containing process b. This is impossible, because a and b are sequential.

Case 4. Network N consists of a subnetwork $N^{\star}_{a}$ containing a in series with a subnetwork $N^{\star\star}_{b}$ containing b. Let $g^{\star}_{i}(t)$ be the density of the completion time of $N^{\star}_{a}$ when factor 1 is at level i. The cdf $G^{\star\star}_{j}(t)$ is defined analogously.

$$G_{ij}(t) = \int_0^t g^{\star}_{i}(x)\, G^{\star\star}_{j}(t-x)\, dx.$$

By Theorem 1, $G^{\star\star}_{2}(t-x) \leqslant G^{\star\star}_{1}(t-x)$ for all $t-x$. By Theorem 2, $g^{\star}_{2}(x) \leqslant g^{\star}_{1}(x)$ for all u, $0 < x < \tau$. Then

$$\begin{aligned} c(t) &= \int_0^t (g^{\star}_{2}(x) - g^{\star}_{1}(x))(G^{\star}_{2}(t-x) - G^{\star}_{1}(t-x))\, dx \\ &\geqslant 0 \end{aligned}$$

for $0 < t < \tau$. Q.E.D.

The reason the inequality is not strict in the theorem above is that certain expressions appearing in products, e.g., $g^{\star}_{2}(x) - g^{\star}_{1}(x)$, may be 0 for an interval of time, leading to a value of 0 for the interaction contrast $c(t)$. If all the relevant densities, cdfs, and differences are not zero throughout an interval, the inequality will be strict.

The proof of the preceding theorem does not apply to OR networks because Theorem 2, used in case 4, does not apply for OR gates (see Appendix). The analogous result for OR networks will be derived below in Theorem 6.

Extended time intervals. The preceding result, Theorem 5, describes the cdf interaction contrast in a sufficiently brief interval after time 0. We now turn to extended time intervals.

We begin with the simplest case. Suppose a network consists only of the two processes a and b in series. Suppose one factor selectively influences a and another factor selectively influences b, each factor producing stochastic dominance. Then for any t,

$$\int_t^\infty c(u)\,du \leqslant 0 \qquad \text{and} \qquad \int_0^t c(u)\,du \geqslant 0, \tag{5}$$

where, as before, $c(t)$ is the cdf interaction contrast.

To see this, note that when the factor influencing a is at level i and the factor influencing b is at level j, the cdf for the completion time is

$$G_{ij}(t) = \int_0^\infty [F_{ai}(a)\, f_{bj}(t-a)]\,da.$$

Then

$$\begin{aligned}
\int_t^\infty c(u)\,du &= \int_t^\infty \int_0^\infty [F_{a1}(a) - F_{a2}(a)][f_{b1}(u-a) - f_{b2}(u-a)]\,da\,du \\
&= \int_0^\infty [F_{a1}(a) - F_{a2}(a)] \int_t^\infty [f_{b1}(u-a) - f_{b2}(u-a)]\,du\,da \\
&= \int_0^\infty [F_{a1}(a) - F_{a2}(a)][\overline{F_{b1}}(t-a) - \overline{F_{b2}}(t-a)]\,da \\
&= \int_0^\infty [F_{a1}(a) - F_{a2}(a)][F_{b2}(t-a) - F_{b1}(t-a)]\,da \\
&\leqslant 0.
\end{aligned}$$

The inequality follows because each factor produces stochastic dominance. The argument for the integral from 0 to t is similar.

The reader may note that because $f_{b1}(u-a) = 0 = f_{b2}(u-a)$ when $a > u$, the region of integration for the double integral above can be expressed as $t \leqslant u < \infty$ and $0 \leqslant a \leqslant u$. That is, the upper limit of the inner integral can be expressed in terms of the outer integrator, u. This does not lead to a problem in interchanging the order of integration, however, because the region of integration does not have to be expressed in this way; the region can simply be expressed as $t \leqslant u < \infty$ and $0 < a < \infty$.

We turn now to extending Inequalities (5) to sequential processes in arbitrary serial-parallel networks. The results rely on a preliminary lemma.

LEMMA 1. Let (a, b) be an interval where a may be $-\infty$ and b may be ∞. Let $H(t)$ and $s(t)$ be functions on (a, b), where s is nonnegative and bounded from above.

(a) If for every $t \in (a, b)$

$$\int_a^t H(x)\,dx \geqslant 0,$$

and if $s(t)$ is monotonically decreasing on (a, b), then for every $t \in (a, b)$

$$\int_a^t H(x)\, s(x)\, dx \geqslant 0.$$

(b) If for every $t \in (a, b)$

$$\int_t^b H(x)\, dx \leqslant 0,$$

and if $s(t)$ is monotonically increasing on (a, b), then for every $t \in (a, b)$

$$\int_t^b H(x)\, s(x)\, dx \leqslant 0.$$

Proof. Note for both (a) and (b) that the existence of the integral of $H(x)$ implies the existence of the integral of $H(x)\, s(x)$ because $0 \leqslant s(x)$ and $s(x)$ is bounded from above.

$$\text{(a)} \quad \int_a^t H(x)\, s(x)\, dx = \int_a^t s(x) \left[\frac{d}{dx} \int_a^x H(u)\, du \right] dx = \int_a^t s(x)\, d \int_a^x H(u)\, du.$$

Integrating by parts,

$$\int_a^t s(x)\, d \int_a^x H(u)\, du = s(t) \int_a^t H(u)\, du - s(a) \int_a^a H(u)\, du - \int_a^t \int_a^x H(u)\, du\, ds(x)$$

(where $s(a)$ is replaced by $\lim_{x \to a} s(x)$ if a is $-\infty$).

On the right hand side, the first term is nonnegative and the second is 0. Because $s(x)$ is monotonically decreasing, $-s(x)$ is monotonically increasing. Then the third term is nonnegative because

$$\int_a^t \int_a^x H(u)\, du\, d[-s(x)] \geqslant \int_a^t 0\, d[-s(x)] = 0.$$

$$\text{(b)} \qquad \int_t^b H(x)\, s(x)\, dx = \int_t^b s(x)\, d \int_x^b -H(u)\, du.$$

The rest of the proof is analogous to that for part (a). Q.E.D.

According to the following theorem, when sequential processes are selectively influenced, the area bounded by the cdf interaction contrast curve is different for AND networks and OR networks. For AND networks, the area bounded by the curve from any time t to infinity is negative or 0. For OR networks, the area from 0 to any time t is positive or 0. Figure 6 illustrates the theorem for the network in

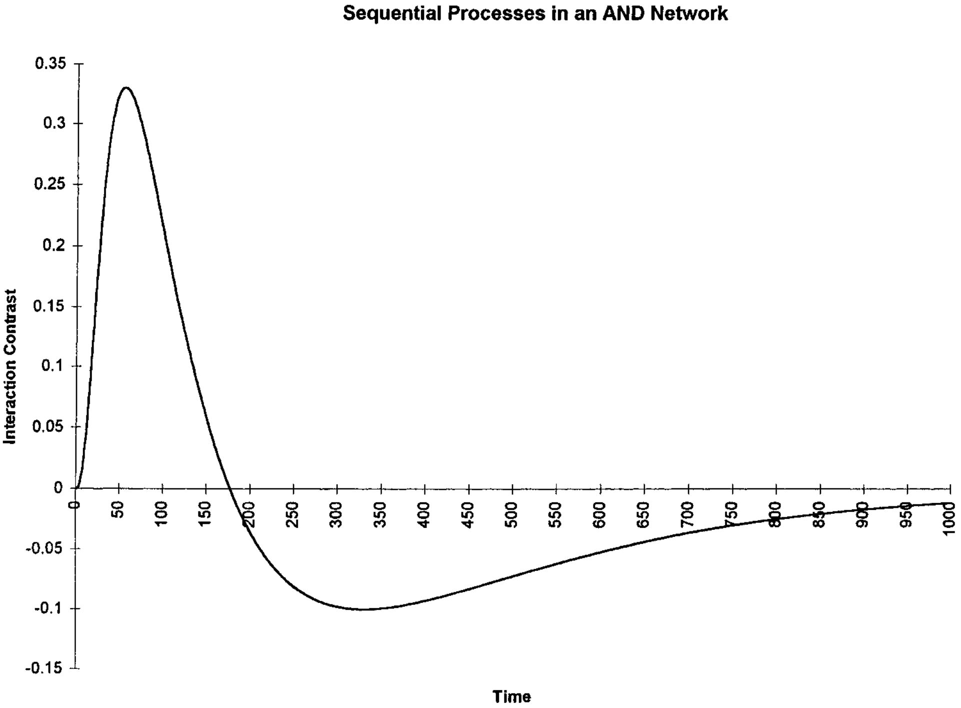

FIG. 6. Interaction contrast function for the sequential processes x and y when the network in Fig. 1 has an AND gate. Process duration means are 10 and 100 for x, 12.5 and 250 for y, and 50 for z. Distributions are as in Fig. 4. The area bounded by the curve from any time t to infinity is less than or equal to 0.

Fig. 1 considered as an AND network. Starting at any time t, the area bounded by the cdf contrast from t to infinity is negative; in particular the area bounded by the entire curve from 0 to infinity equals -13.0. Figure 7 is the corresponding illustration when the network is considered an OR network. The area bounded by the cdf contrast from 0 to any time t is positive; in particular, the area bounded by the entire curve from 0 to infinity equals 23.0.

THEOREM 6. *Suppose a and b are sequential processes in an independent series-parallel network. Suppose one factor selectively influences a and another selectively influences b, producing stochastic dominance. Let $c(t)$ be the cdf interaction contrast for a and b. If all the gates are AND gates, then for all $t \geqslant 0$*

$$\int_t^{\infty} c(u)\, du \leqslant 0.$$

If all the gates are OR gates, then for all $t \geqslant 0$

$$\int_0^{t} c(u)\, du \geqslant 0.$$

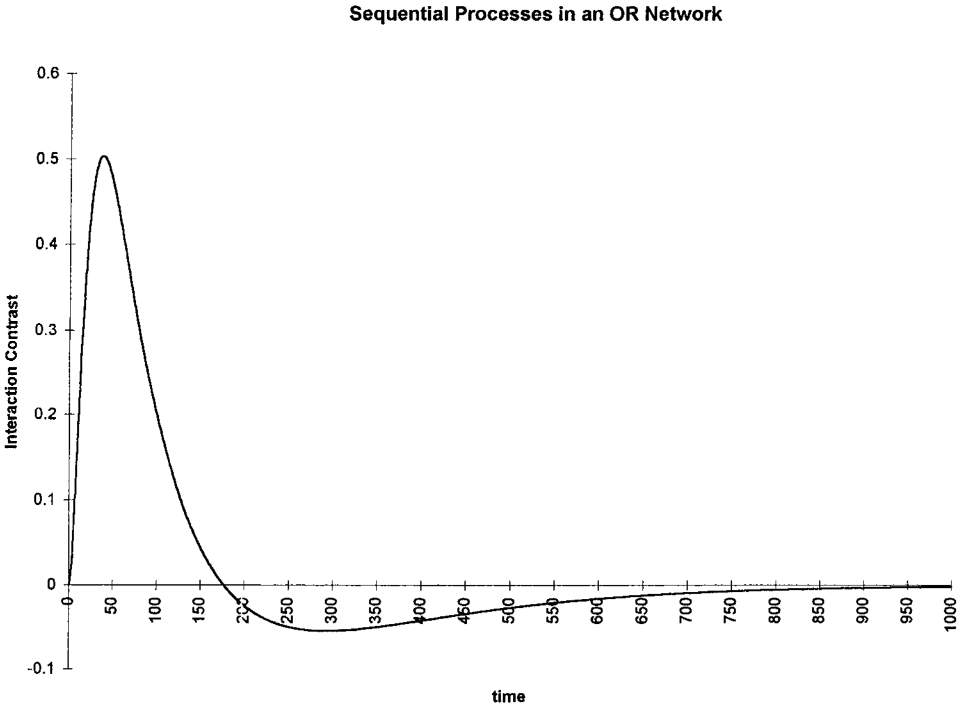

FIG. 7. Interaction contrast function for the sequential processes x and y when the network in Fig. 1 has an OR gate. Process duration means are 10 and 100 for x, 12.5 and 250 for y, and 500 for z. Distributions are as in Fig. 4. The area bounded by the curve from 0 to any time t greater than or equal to 0.

Proof. The proof is by induction on the number of processes.

If a and b are the only processes in the network the conclusion is in the earlier Inequalities (5).

Suppose the conclusion is true for all independent serial-parallel networks with n or fewer processes, and consider an independent serial-parallel network N with $n+1$ processes.

Case 1. Suppose N consists of a subnetwork $N^{\star}_{ab}$ with n or fewer processes, including both a and b, and another subnetwork $N^{\star\star}$ in parallel with subnetwork $N^{\star}_{ab}$. Let $G^{\star}_{ij}(t)$ be the cdf of the completion time for $N^{\star}_{ab}$, when the factor influencing a is at level i and the factor influencing b is at level j. Let $G^{\star\star}(t)$ be the completion time for subnetwork $N^{\star\star}$.

If all the gates are AND gates, the cumulative distribution function for the network N itself is $G_{ij}(t) = G^{\star}_{ij}(t) \cdot G^{\star\star}(t)$. Let $c^{\star}(t) = G^{\star}_{22}(t) - G^{\star}_{21}(t) - G^{\star}_{12}(t) + G^{\star}_{11}(t)$. By the induction hypothesis,

$$\int_t^{\infty} c^{\star}(u)\, du \leqslant 0.$$

Then by Lemma 1,

$$\int_t^\infty c(u)\,du = \int_t^\infty [G^\star_{22}(u) - G^\star_{21}(u) - G^\star_{12}(u) + G^\star_{11}(u)]\, G^{\star\star}(u)\,du$$

$$= \int_t^\infty c^\star(u)\, G^{\star\star}(u)\,du \leqslant 0.$$

If all the gates are OR gates, the cumulative distribution function for the network itself is $G_{ij}(t) = G^\star_{ij}(t) + G^{\star\star}(t) - G^\star_{ij}(t)\, G^{\star\star}(t)$, and the proof is similar to that for AND gates.

Case 2. Suppose N consists of a subnetwork $N^\star_{ab}$ with n or fewer processes, including both a and b, and a subnetwork $N^{\star\star}$ in series with subnetwork $N^\star_{ab}$. Let $G^\star_{ij}(t)$ be the cumulative distribution function of the completion time for $N^\star_{ab}$, when the first factor is at level i and the second is at level j. Let $g^{\star\star}(t)$ be the density for the completion time of $N^{\star\star}$.

Then

$$G_{ij}(t) = \int_0^\infty G^\star_{ij}(t-z)\, g^{\star\star}(z)\,dz.$$

If the gates are AND gates,

$$\int_t^\infty c(u)\,du = \int_t^\infty \int_0^\infty c^\star(u-z)\, g^{\star\star}(z)\,dz\,du$$

$$= \int_0^\infty \int_t^\infty c^\star(u-z)\,du\; g^{\star\star}(z)\,dz.$$

By a transformation of variables, the inner integral can be written

$$\int_{t-z}^\infty c^\star(v)\,dv.$$

The inner integral is less than or equal to 0 by the induction hypothesis for any $t-z$. Then

$$\int_t^\infty c(u)\,du \leqslant 0.$$

If the gates are OR gates, the argument is similar.

Case 3. Suppose N consists of a subnetwork containing a in parallel with a subnetwork containing b. This is impossible because a and b are sequential processes.

Case 4. Suppose N consists of a subnetwork N_a containing a in series with a subnetwork N_b containing b. The reasoning is analogous to the case of two processes in series. Let the cdf of the completion time of N_a be $G^{\star}_{ai}(t)$, when the first factor is at level i, and let $G^{\star}_{bj}(t)$ be defined analogously. By Theorem 1, $G_{a1}(t) \geqslant G_{a2}(t)$ because $F_{a1}(t) \geqslant F_{a2}(t)$, and $G_{b1}(t) \geqslant G_{b2}(t)$ because $F_{b1}(t) \geqslant F_{b2}(t)$. The conclusion follows from the reasoning leading to Inequalities (5), with $G_{ai}(t)$ substituted for $F_{ai}(t)$, $i = 1, 2$, and $G_{bj}(t)$ substituted for $F_{bj}(t)$, $j = 1, 2$, respectively. Q.E.D.

Two immediate consequences of Theorem 6 are worth noting. First, when the assumptions of Theorem 6 are met, an earlier result from Theorem 5 for AND networks can be reformulated for OR networks. That is, for an OR network meeting the assumptions of Theorem 6, it is easily seen that there is no time τ such that $c(t)$ is negative for all times t in the interval $(0, \tau)$. This result, based on stronger assumptions than those of Theorem 5, circumvents the difficulty mentioned earlier in proving Theorem 5 for OR networks.

Second, results for mean interaction contrasts can be obtained from the results of Theorem 6 for cdf interaction contrasts by evaluating the integrals in Theorem 6 from 0 to ∞. The interaction contrast for mean response times is

$$E[RT_{22}] - E[RT_{12}] - E[RT_{21}] + E[RT_{11}].$$

From Eq. (4) and Theorem 6, it follows immediately that the mean interaction contrast for sequential processes is positive or zero for AND networks and negative or zero for OR networks. This provides an alternative proof, based on different assumptions, of earlier results by Schweickert and Townsend (1989), Schweickert, Fisher, and Goldstein (1992), and Schweickert and Wang (1993).

DISCUSSION

To summarize, when two processes in a serial-parallel network are selectively influenced by experimental factors, the cdf interaction contrasts are informative about whether the processes are concurrent or sequential and whether the network has AND or OR gates. The details are in the theorems themselves, but the main ideas can be stated simply. The main assumptions are (a) the processes involved in performing a task are arranged in a directed acyclic network in which all the gates are AND gates or all the gates are OR gates, (b) the network is a serial-parallel network, (c) there are two experimental factors which selectively influence two different processes, (d) when the level of a factor is changed, the corresponding cdfs for the duration of the influenced process exhibit stochastic dominance, and (e) the process durations are mutually independent random variables at all levels of the two factors. The main results are the following.

(a) If the two selectively influenced processes are concurrent, then the cdf interaction contrast will have the same sign at all times t, nonnegative for AND gates and nonpositive for OR gates.

(b) If the two processes are sequential, then the cdf interaction contrast is nonnegative for a period of time after 0, and then, for an AND network, changes sign at least once.

(c) Suppose the two processes are sequential. If the network has OR gates, then the area bounded by the cdf interaction contrast and the time axis will be nonnegative from time 0 to any arbitrary time t. If the network has AND gates, then the bounded area will be nonpositive from any arbitrary time t to ∞.

Our results do not imply that the cdf interaction contrast changes signs for sequential processes in an OR network. Consequently, the pattern for sequential processes in an OR network may be the same as that for concurrent processes in an AND network. These two arrangements are different enough that practical knowledge about the experimental factors would ordinarily distinguish between them. Aside from these two cases, for serial-parallel networks, one can distinguish concurrent from sequential processes and OR networks from AND networks with the cdf interaction contrast.

An overview of our main results can be obtained by comparing the cdf interaction contrast functions in Figs. 4–7. If data in the form of one of these figures were produced by selectively influencing processes in an independent serial-parallel AND or OR network, the investigator could classify the underlying process arrangement in the following way.

In Figs. 4 and 5, the interaction contrast function does not change sign. This is predicted when two concurrent processes are selectively influenced. The positive function in Fig. 4 is predicted by concurrent processes in an AND network. Our results do not rule out the possibility that the function in Fig. 4 was produced by sequential processes in an OR network. The negative function in Fig. 5 is predicted by concurrent processes in an OR network. In Figs. 6 and 7, the interaction contrast function changes sign. This sign change cannot be produced by factors selectively influencing concurrent processes, but can be predicted by factors selectively influencing sequential processes. The area bounded by the cdf contrast and the horizontal axis is informative. The net area from any time t to ∞ is negative in Fig. 6, characteristic of sequential processes in an AND network. The net area from 0 to any time t is positive in Fig. 7, characteristic of sequential processes in an OR network.

The appearance of a predicted pattern in the data does not imply the presence of the particular process arrangement leading to the prediction, of course, because a completely different architecture may lead to the same prediction. However, when a process arrangement predicts a certain pattern in the data, and a different pattern appears, then the process arrangement can be eliminated as a possibility. The usefulness of the predicted patterns is not so much that they are difficult to mimic as that they are easily checked.

APPENDIX

Theorem 2 and OR Networks

It might seem that if the assumptions of Theorem 2 were true for an OR network, the conclusion would necessarily follow, but it does not, as the following example shows. Consider the simple OR network with a process a in parallel with a process z. When the factor is at level i, the survivor function is

$$\bar{G}_i(t) = \overline{F_{ai}}(t)\,\overline{F_z}(t).$$

Then

$$g_1(t) - g_2(t) = [f_{a1}(t) - f_{a2}(t)]\,\overline{F_z}(t) + [\overline{F_{a1}}(t) - \overline{F_{a2}}(t)]\,f_z(t).$$

By the assumptions of Theorem 2, $f_{a1}(t) \geqslant f_{a2}(t)$ for $t < \tau$. If (at least a weak version of) Theorem 2 holds, then in some vicinity of 0 it must be true that

$$[f_{a1}(t) - f_{a2}(t)]\,\overline{F_z}(t) + [\overline{F_{a1}}(t) - \overline{F_{a2}}(t)]\,f_z(t) \geqslant 0.$$

Since $\overline{F_z}(t) = 1 - F_z(\mathrm{t}) \geqslant 0$ and (for $t < \tau$) $F_{a1}(t) - F_{a2}(\mathrm{t}) \geqslant 0$, this inequality is equivalent to

$$\frac{f_{a1}(t) - f_{a2}(t)}{F_{a1}(\mathrm{t}) - F_{a2}(t)} \geqslant \frac{f_z(t)}{1 - F_z(t)}.$$

It is easy to see that the left-hand side of this inequality,

$$h(t) = \frac{f_{a1}(t) - f_{a2}(t)}{F_{a1}(t) - F_{a2}(t)} = \log'[F_{a1}(t) - F_{a2}(t)],$$

tends to infinity as $t \to 0$, because $\log[F_{a1}(t) - F_{a2}(t)] \to -\infty$. We now construct the density function $f_z(t)$ in the following way (refer to Fig. 8). Between points $t = 0$ and $t = 1$ the function $f_z(t)$ is zero except within intervals $2^{-n} + \delta_n/2$ ($n = 0, 1, 2, \ldots$) where its value is $h(2^{-n})\,k$, $k > 1$ being an arbitrary constant. As a result, $f_z(t)$ exceeds $h(t)$between $t = 0$ and $t = 1$ within each of these intervals. The width of the intervals $\delta_n$is computed as

$$\delta_n = \frac{\varepsilon_n}{h(2^{-n})\,k},$$

where ε_n is any sequence of positive numbers such that

$$\varepsilon_n < (2^{-n} - 2^{-n-1})\,h(2^{-n})\,k$$

and

$$\sum_1^{\infty} \varepsilon_n = p < 1.$$

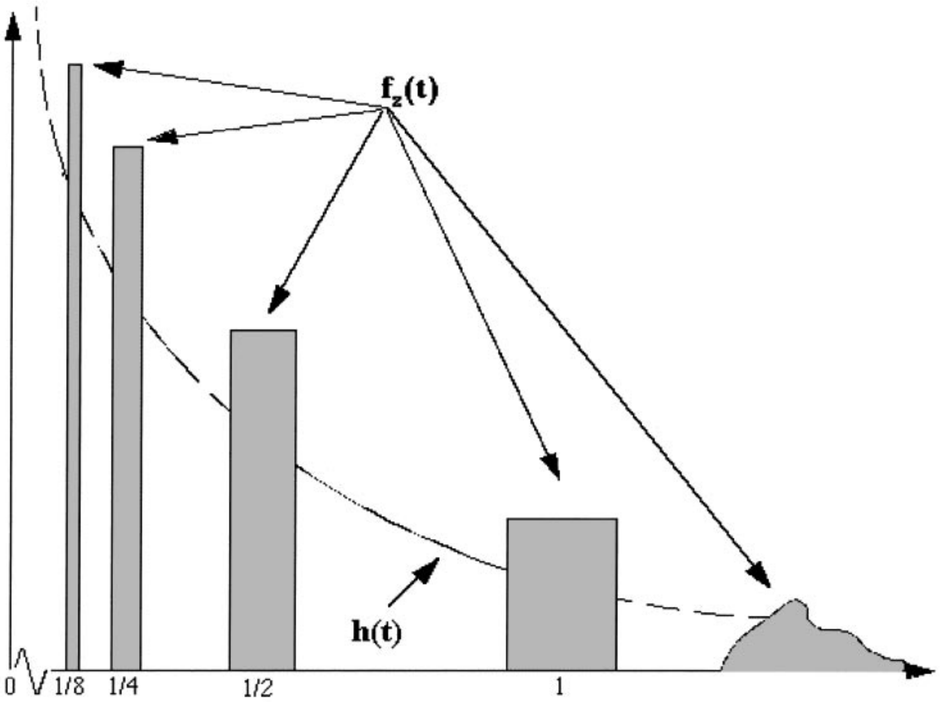

FIG. 8. The density function for process z is higher than the function $h(t)$ in an infinite number of places. Details in text.

It is easy to check that thus defined intervals do not overlap and that the area under $f_z(t)$ within any interval $2^{-n} \pm \delta_n/2$ equals ε_n. To the right of $t=1$ the function $f_z(t)$ can be extended arbitrarily except that $f_z(t) \geqslant 0$ and

$$\int_1^{\infty} f_z(t)\,dt = 1 - p.$$

The total area under $f_z(\mathrm{t})$ therefore is

$$\sum_1^{\infty} \varepsilon_n + \int_1^{\infty} f_z(t)\,dt = 1,$$

and $f_z(t)$ is a legitimate density function. Observe now that any vicinity of $t=0$, however small, contains an infinity of intervals $2^{-n} \pm \delta_n/2$, and within each of these intervals $f_z(t)$ exceeds $h(t)$ by a factor of $k>1$. Since $h(t) \to \infty$ as $t \to 0$, the difference $f_z(t) - h(t)$ increases as $t \to 0$. At the same time as $t \to 0$ the difference between $f_z(t)$ and $f_z(t)/[1 - F_z(t)]$ tends to zero. We conclude that in any sufficiently small vicinity of $t=0$ the inequality

$$\frac{f_{a1}(\mathrm{t}) - f_{a2}(t)}{F_{a1}(t) - F_{a2}(t)} \geqslant \frac{f_z(t)}{1 - F_z(t)}$$

will be violated an infinite number of times. This example shows, then, that Theorem 2 is not generally true for independent OR networks.

Another approach. Theorem 2 describes the behavior of densities at values of t below some critical value. At first it might seem that a similar result describing the behavior of densities at values of t exceeding some critical value would apply for OR gates. It might seem, in other words, that a version of Theorem 2 would apply if the direction of every inequality in the conclusion was reversed.

The following counterexample shows this to be false for two processes in series, so it is false for more complicated networks with OR gates also. Suppose a and z are in series. Suppose for all $t>\tau$, $f_{a1}(t)-f_{a2}(t)\geqslant 0$. We show it is possible nevertheless that

$$g_1(t)-g_2(t)=\int_0^t [f_{a1}(x)-f_{a2}(x)]\, f_z(t-x)\, dx \leqslant 0$$

for all $t>\tau$.

Let $f_z(t)=e^{-t}$. Then

$$\int_0^t [f_{a1}(x)-f_{a2}(x)]\, f_z(t-x)\, dx = \int_0^t [f_{a1}(x)-f_{a2}(u)]\, e^{x-t}\, dx$$

$$= e^{-t}\int_0^t [f_{a1}(x)-f_{a2}(x)]\, e^x\, dx.$$

Since $f_{a1}(t)-f_{a2}(t)\geqslant 0$ above τ,

$$e^{-t}\int_t^\infty [f_{a1}(x)-f_{a2}(x)]\, e^x\, dx \geqslant 0,$$

and therefore

$$e^{-t}\int_0^t [f_{a1}(x)-f_{a2}(x)]\, e^x\, dx \leqslant e^{-t}\int_0^\infty [f_{a1}(x)-f_{a2}(x)]\, e^x\, dx$$

$$= e^{-t}\left\{\int_0^\infty f_{a1}(x)\, e^x\, dx - \int_0^\infty f_{a2}(x)\, e^x dx\right\}.$$

To conclude the counterexample it suffices to show that the densities $f_{a1}(t)$ and $f_{a2}(\mathrm{t})$ can be chosen so that

$$\int_0^\infty f_{a1}(t)\, e^t\, dt = \int_0^\infty f_{a2}(t)\, e^t\, dt,$$

while $f_{a1}(t)$ exceeds $f_{a2}(\mathrm{t})$ for all sufficiently large t. As an example, the exponential function $f_{a1}(t)=(4/3)\, e^{-4t/3}$ exceeds the two-stage gamma function $f_{a2}(t)=4te^{-2t}$ at sufficiently large values of t, but they both have the same expected value of e^t, equal to 4.

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