Context-Dependence and Contextuality

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Contextuality: My approach

(quantum) contextuality

general probabilistic issue

classical (Kolmogorovian) probability

with contextual labeling
Contextuality-by-Default Theory: Three Forms of Context-Dependence
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Contextuality-by-Default Theory: Three Forms of Context-Dependence

2. Direct (cross-) influences (inconsistent connectedness). Can be present or absent.
Contextuality-by-Default Theory: Three Forms of Context-Dependence

2. Direct (cross-) influences (inconsistent connectedness). Can be present or absent.
3. Contextuality proper. Can be present or absent.
1. Random variables are contextually labeled (i.e., their context is part of their identity).

\[ R_1^1 \quad R_2^1 \quad . \quad R_4^1 \quad c_1 \]

\[ R_1^2 \quad R_2^2 \quad R_3^2 \quad . \quad c_2 \]

\[ . \quad R_2^3 \quad R_3^3 \quad R_4^3 \quad c_3 \]

\[ q_1 \quad q_2 \quad q_3 \quad q_4 \quad \mathcal{R} \]
1. Random variables are contextually labeled (i.e., their context is part of their identity).

This implies that the sets of random variables in different contexts are disjoint.
2. The distributions of two connected random variables (measuring the same content in different contexts) may be different.
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\[ R_1^1 \cdot R_2^1 \cdot R_4^1 c_1 \]
\[ R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot R_5^2 c_2 \]
\[ \cdot R_2^3 \cdot R_3^3 \cdot R_4^3 c_3 \]
\[ q_1 q_2 q_3 q_4 R \]
2. The distributions of two connected random variables (measuring the same content in different contexts) may be different.

\[
\begin{array}{cccc}
R^1_1 & R^1_2 & \cdot & R^1_4 \\
R^2_1 & R^2_2 & R^2_3 & \cdot \\
\cdot & R^3_2 & R^3_3 & R^3_4 \\
q_1 & q_2 & q_3 & q_4 \\
\end{array}
\]

The interpretation is: “Direct” influences (cross-influences), Signaling, Disturbance, Context-dependent biases, etc.
3. Contextuality is present if the joint distributions within contexts are incompatible with certain joint distributions imposed on the content-sharing random variables.
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3. Contextuality is incompatibility of joint distributions of the bunched random variables with certain joint distributions imposed on the content-sharing random variables.
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Contextuality-by-Default Theory: Contextuality proper

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Contextuality-by-Default Theory: Contextuality proper

3. Contextuality is incompatibility of joint distributions of the **bunched** random variables with certain **couplings** imposed on the **connected** random variables.

<table>
<thead>
<tr>
<th>R₁₁</th>
<th>R₂₁</th>
<th>.</th>
<th>R₄₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>R₁₂</td>
<td>R₂₂</td>
<td>R₃₂</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>R₂₃</td>
<td>R₃₃</td>
<td>R₄₃</td>
</tr>
<tr>
<td>q₁</td>
<td>q₂</td>
<td>q₃</td>
<td>q₄</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c₁</th>
<th>T₁¹</th>
<th>T₂¹</th>
<th>.</th>
<th>T₄¹</th>
</tr>
</thead>
<tbody>
<tr>
<td>c₂</td>
<td>T₁²</td>
<td>T₂²</td>
<td>T₃²</td>
<td>.</td>
</tr>
<tr>
<td>c₃</td>
<td>T₃²</td>
<td>T₄²</td>
<td>T₄³</td>
<td>.</td>
</tr>
</tbody>
</table>

vs

| q₁  | q₂  | q₃ | q₄ |

[J]
3. Contextuality is incompatibility of joint distributions of the bunched random variables with certain couplings imposed on the connected random variables.

\[
\begin{array}{cccc}
S_1^1 & S_2^1 & \cdot & S_4^1 \\
S_1^2 & S_2^2 & S_3^2 & \cdot \\
\cdot & S_2^3 & S_3^3 & S_4^3 \\
q_1 & q_2 & q_3 & q_4
\end{array}
\]

\[
\begin{array}{cccc}
\cdot & S_2^1 & S_3^1 & \cdot \\
S_2^2 & S_3^2 & S_4^2 & \cdot \\
\cdot & \cdot & S_3^3 & S_4^3 \\
q_1 & \cdot & q_2 & q_3
\end{array}
\]

\[
\begin{array}{cccc}
S_3^1 & S_3^2 & \cdot & S_4^1 \\
S_3^2 & S_3^3 & S_3^4 & \cdot \\
q_1 & q_2 & q_3 & q_4
\end{array}
\]
3. Contextuality is incompatibility of joint distributions of the bunched random variables with certain couplings imposed on the connected random variables.

![Table]

There is a universal measure of contextuality involving signed measures.
3. Contextuality is incompatibility of joint distributions of the bunched random variables with certain couplings imposed on the connected random variables.

There is a universal measure of contextuality involving signed measures.
### Contextuality-by-Default Theory: Multimaximal Couplings

Couplings are imposed so that any two connected random variables are equal to each other with maximal possible probability. This is called a multi-maximal coupling. (Can be generalized.)
Contextuality-by-Default Theory: Multimaximal Couplings

Couplings are imposed so that any two connected random variables are equal to each other with maximal possible probability. This is called a multi-maximal coupling. *(Can be generalized.*)
Necessity of Contextual Labeling: Traditional View is Contradictory

Noncontextual labeling is not an option, even in the absence of direct influences:

\[
\begin{array}{cccc}
R_1 & R_2 & \cdot & R_4 \\
R_1 & R_2 & R_3 & \cdot \\
\cdot & R_2 & R_3 & R_4 \\
\cdot & R_2 & R_3 & R_4 \\
\end{array}
\]
(Non)Contextuality may exist with or without direct influences.
(Non)Contextuality may exist with or without direct influences.

Direct influences are causal, contextuality is correlational:
(Non)Contextuality may exist with or without direct influences.

Direct influences are causal, contextuality is correlational:

1. Direct influences are manifested in changes of marginal distributions, contextuality is revealed on the level of joint distributions;
Contextuality versus Inconsistent Connectedness

1. (Non)Contextuality may exist with or without direct influences.

2. Direct influences are causal, contextuality is correlational:
   - direct influences are manifested in changes of marginal distributions, contextuality is revealed on the level of joint distributions;
   - contextuality can relate spacelike-separated measurements;
(Non)Contextuality may exist with or without direct influences.

Direct influences are causal, contextuality is correlational:

1. Direct influences are manifested in changes of marginal distributions, contextuality is revealed on the level of joint distributions;
2. Contextuality can relate spacelike-separated measurements;
3. With timelike separation, a future measurement can create an effective context for a past one.
Couplings are imposed so that any two connected random variables are equal to each other with maximal possible probability. This is called a multi-maximal coupling. (Can be generalized.)
In a well-designed theory a multi-maximal coupling always exists and is unique. This is true if all random variables are binary (dichotomous). This is not generally true.
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In a well-designed theory a multi-maximal coupling always exists and is unique.

1. This is true if all random variables are binary (dichotomous).
2. This is not generally true.
Example: Cyclic systems of binary random variables

\[
\begin{array}{cccccccccc}
R_1^1 & R_1^2 & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & c_1 \\
\cdot & R_2^2 & R_3^2 & \cdot & \cdots & \cdot & \cdot & \cdot & c_2 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & c_{n-1} \\
\cdot & \cdot & \cdot & \cdot & \cdots & R_{n-1}^{n-1} & R_n^{n-1} & c_n \\
R_1^n & \cdot & \cdot & \cdot & \cdots & \cdot & R_n^n & c_{\text{CYC}_n} \\
q_1 & q_2 & q_3 & q_4 & \cdots & q_{n-1} & q_n
\end{array}
\]
Example: Cyclic systems of binary random variables

- \( n > 5 \) investigated in psychophysics (Ru Zhang, Cervantes)
- \( n = 5 \) — Klyachko-Can-Binicoglu-Shumovsky-type system
- \( n = 4 \) — Einstein-Podolsky-Rosen/Bohm-Bell-type system
- \( n = 3 \) — Suppes-Zanotti-Leggett-Garg-type system
- \( n = 2 \) — question order (Moore-Wang-Busemeyer) type system
Example: Cyclic systems of binary random variables

**Theorem**

A cyclic system of binary random variables is contextual if and only if

$$\max_{(\tau_1, \ldots, \tau_k) \in \{-1, 1\}^n : \prod_{i=1}^n \tau_i = -1} \sum_{i=1}^n \tau_i \langle R_i^i R_i^{i\oplus 1} \rangle > n - 2 + \sum_{i=1}^n \left| \langle R_i^i \rangle - \langle R_i^{i\oplus 1} \rangle \right|.$$
Flow Chart of Contextuality Analysis
Flow Chart of Contextuality Analysis

empirical measurements → initial system of rvs

expanded system of rvs → canonical (split) representation

contextuality analysis
Flow Chart of Contextuality Analysis

ontic-epistemic interplay: the representation is non-unique (in many ways).
Flow Chart of Contextuality Analysis: Expansion through Joining

initial system of rvs → expanded system of rvs

Ontic-epistemic interplay: joining is non-unique and selective. There are cases (e.g., question-order effect) when it is unwarranted.
Flow Chart of Contextuality Analysis: Expansion through Joining

Joining:

\[
\begin{align*}
R_1^1 & \quad R_2^1 & \quad \cdot & \quad c = 1 \\
R_1^2 & \quad R_2^2 & \quad \cdot & \quad 2 \\
R_1^3 & \quad \cdot & \quad R_3^3 & \quad \cdot & \quad 3 \\
q = 1 & \quad 2 & \quad 3
\end{align*}
\]

\[
\begin{align*}
R_1^1 & \quad R_2^1 & \quad \cdot & \quad (R_1^1, R_2^1) & \quad c = 1 \\
R_1^2 & \quad R_2^2 & \quad \cdot & \quad (R_1^2, R_2^2) & \quad 2 \\
R_1^3 & \quad \cdot & \quad R_3^3 & \quad \cdot & \quad 3 \\
q = 1 & \quad 2 & \quad 3 & \quad 12 & \quad R^*
\end{align*}
\]

Ontic-epistemic interplay: joining is non-unique and selective. There are cases (e.g., question-order effect) when it is unwarranted.
Flow Chart of Contextuality Analysis: Expansion through Joining

Joining:

\[ R_1^1 \quad R_2^1 \quad \cdot \quad c = 1 \]
\[ R_1^2 \quad R_2^2 \quad \cdot \quad 2 \]
\[ R_1^3 \quad \cdot \quad R_3^3 \quad \cdot \quad q = 1 \quad 2 \quad 3 \]

\[ R_1^1 \quad R_2^1 \quad \cdot \quad (R_1^1, R_2^1) \quad c = 1 \]
\[ R_1^2 \quad R_2^2 \quad \cdot \quad (R_1^2, R_2^2) \quad 2 \]
\[ R_1^3 \quad \cdot \quad R_3^3 \quad \cdot \quad q = 1 \quad 2 \quad 3 \quad 12 \quad (R_1^3, R_3^3) \quad R^* \]

Ontic-epistemic interplay: joining is non-unique and selective. There are cases (e.g., question-order effect) when it is unwarranted.
Flow Chart of Contextuality Analysis: Expansion through Coarsening

Coarsening (coarse-graining):

\[ R_1 \rightarrow q_1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 = R_1 \rightarrow 2, 3, 4, 5 \rightarrow \cdots \rightarrow R_1, 2 \rightarrow 3, 5 \rightarrow 4 \rightarrow \cdots \]

Ontic-epistemic interplay: coarsening is non-unique and selective. Not all possible coarsenings may be of interest (depends on the internal structure, e.g., linear ordering or metric, of the space of possible values).
Flow Chart of Contextuality Analysis: Expansion through Coarsening

Initial system of rvs

Coarsening (coarse-graining):

Expanded system of rvs

Ontic-epistemic interplay: coarsening is non-unique and selective. Not all possible coarsenings may be of interest (depends on the internal structure, e.g., linear ordering or metric, of the space of possible values).
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Flow Chart of Contextuality Analysis: Expansion through Coarsening

Coarsening (coarse-graining):

\[ R^1_q \]

\[
\begin{array}{c|c|c|c|c|c}
1 & 1 & 1, 2, 3, 4, 5 & \cdots & 1, 2, 3, 4, 5 & \cdots \\
2 & 2 & \cdots & 4, 5 & \cdots & \cdots \\
3 & 3 & & 5 & & \\
4 & 4 & & & & \\
5 & 5 & & & & \\
\end{array}
\]

Ontic-epistemic interplay: coarsening is non-unique and selective. Not all possible coarsenings may be of interest (depends on the internal structure, e.g., linear ordering or metric, of the space of possible values).
Flow Chart of Contextuality Analysis: Canonical Representation

- expanded system of rvs
- canonical (split) representation

Each random variable in the expanded system is replaced with its split representation, a binary random variable isolating one of the values from the rest (detecting this value).
Flow Chart of Contextuality Analysis: Canonical Representation

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Flow Chart of Contextuality Analysis: Canonical Representation

\[ R_1^1 \quad c = 1 \]
\[ R_1^2 \quad c = 2 \]
\[ q = 1 \]
\[ \mathcal{R} \]

\[ D^1_{W1} \quad D^1_{W2} \quad D^1_{W(2^{k-1} - 1)} \quad c = 1 \]
\[ D^2_{W1} \quad D^2_{W2} \quad \cdots \quad D^2_{W(2^{k-1} - 1)} \quad c = 2 \]
\[ q = W1 \quad W2 \quad \cdots \quad W(2^{k-1} - 1) \]

The support of the system does not change.
Flow Chart of Contextuality Analysis: Canonical Representation

The support of the system does not change.
Theorem

The system $\mathcal{D}$ is noncontextual if and only if one of the $R_1^1$ and $R_2^2$ nominally dominates the other.
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**Definition**

$R_1^1$ nominally dominates $R_1^2$ if $Pr[R_1^1 = i] < Pr[R_1^2 = i]$ for no more than one value of $i = 1, \ldots, k$. 
Flow Chart of Contextuality Analysis: Canonical Representation

Theorem

The system \( \mathcal{D} \) is noncontextual if and only if one of the \( R_1^1 \) and \( R_1^2 \) nominally dominates the other.

Example

<table>
<thead>
<tr>
<th>x:</th>
<th>( R_1^1 = x )</th>
<th>( R_1^2 = x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>.2</td>
<td>.3</td>
<td>.1</td>
</tr>
<tr>
<td>.1</td>
<td>.2</td>
<td>.4</td>
</tr>
</tbody>
</table>
Flow Chart of Contextuality Analysis: Canonical Representation

The system $\mathcal{D}$ is noncontextual if and only if one of the $R_1^1$ and $R_1^2$ nominally dominates the other.

Example

<table>
<thead>
<tr>
<th>$x$:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pr[R_1^1 = x]$</td>
<td>.2</td>
<td>.3</td>
<td>.1</td>
<td>.3</td>
<td>.1</td>
</tr>
<tr>
<td>$\Pr[R_1^2 = x]$</td>
<td>.1</td>
<td>.2</td>
<td>.4</td>
<td>.3</td>
<td>0</td>
</tr>
</tbody>
</table>
Measure of contextuality: Quasi-couplings

$\mathbf{D}_1 \quad \mathbf{D}_2 \quad \cdot \quad \mathbf{D}_4$

$\begin{array}{ccc}
\mathbf{D}_1^2 & \mathbf{D}_2^2 & \cdot \\
\cdot & \mathbf{D}_2^3 & \mathbf{D}_3^3 \\
\cdot & \cdot & \mathbf{D}_4^3
\end{array}$

$\rightarrow$

$\begin{array}{ccc}
\mathbf{S}_1^1 & \mathbf{S}_2^1 & \mathbf{S}_4^1 \\
\mathbf{S}_1^2 & \mathbf{S}_2^2 & \mathbf{S}_3^2 \\
\cdot & \mathbf{S}_2^3 & \mathbf{S}_3^3 \\
\cdot & \cdot & \mathbf{S}_4^3
\end{array}$

TOTAL VARIATION:

$$\sum | p( s_1^1, s_2^1, s_4^1, s_1^2, s_2^2, s_3^3, s_3^3, s_4^3 ) | = V_T \geq 1$$

$V_T = 1$ if and only if the quasi-coupling is a proper coupling
Measure of contextuality

Theorem

Any system $\mathcal{R}$ of random variables has a quasi-coupling $S$ such that
(i) any pair $\left( S^c_q, S'^c_q \right)$ is properly distributed, and the probability of $S^c_q = S'^c_q$ has the maximal possible value;
(ii) the value of total variation $V_T$ in $S$ has the smallest possible value among all quasi-couplings satisfying (i).
Measure of contextuality

Theorem

Any system $\mathcal{R}$ of random variables has a quasi-coupling $S$ such that

(i) any pair $(S^c_q, S^{c'}_q)$ is properly distributed, and the probability of $S^c_q = S^{c'}_q$ has the maximal possible value;

(ii) the value of total variation $V_T$ in $S$ has the smallest possible value among all quasi-couplings satisfying (i).

The minimum value of $V_T - 1$ can be taken as a universal measure of contextuality in any system of random variables.

Its value is 0 if and only if the system is not contextual.
Theorem

Contextuality of a system does not change if every context-content pair associated with no measurement is assigned a deterministic random variable ("dummy measurement").
**Dummy measurements**

\[
\begin{array}{cccc}
D_1^1 & D_2^1 & \cdot & D_4^1 \\
D_1^2 & D_2^2 & D_3^2 & \cdot \\
\cdot & D_2^3 & D_3^3 & D_4^3 \\
\end{array}
\quad \mapsto \quad
\begin{array}{cccc}
D_1^1 & D_2^1 & C_3^1 & D_4^1 \\
D_1^2 & D_2^2 & D_3^2 & C_3^2 \\
C_1^3 & D_2^3 & D_3^3 & D_4^3 \\
\end{array}
\]

**Theorem**

*Contextuality of a system does not change if every context-content pair associated with no measurement is assigned a deterministic random variable (“dummy measurement”).*