A Proof of Specker's Principle

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Introduction:

Specker's principle: pairwise orthogonal propositions are jointly orthogonal.

Powerful in the quest for characterisations of QM (cf. e.g. Cabello's derivation of the quantum bound for the Klyachko inequality).

But elusive:

> Ernst and I spent many hours discussing the principle [...]. The difficulty lies in trying to justify it on general physical grounds, without already assuming the Hilbert space formalism of quantum mechanics. We decided to incorporate the principle as an axiom in our definition of partial Boolean algebras [...].
> I have never found a general physical justification for [it]

(Kochen, as reported by Cabello).

Today: a proof of Specker's principle from the assumption of maximally entangled states together with no-signalling (!).

(Cf. Popescu and Rohrlich: Bell non-locality and no-signalling as axioms for QM?)

Structure of the talk:

- I first tell a variant of Specker's 1960 fable of the seer of Nineveh.
- Then I use it to construct a general proof of Specker's principle.
- I conclude with some discussion and open questions.

Specker's seer meets Popescu and Rohrlich

At the Assyrian school for prophets in Arba'ilu, there taught, in the age of king Asarhaddon, a sage from Nineveh. He was an outstanding representative of his discipline (solar and lunar eclipses), who, except for the heavenly bodies, had thoughts almost only for his *two* daughters. His teaching success was modest; the discipline was seen as dry, and furthermore required previous mathematical knowledge that was rarely to be found. If in his teaching he thus failed to capture the interest he would have wanted from the students, he won it overabundantly in a different field: no sooner had his daughters reached the marriageable age, than he was flooded with requests for their hand from students and young graduates. And even though he did not imagine wishing to keep them with him forever, yet they were still far too young, and the suitors in no way worthy of them. And so that they should be themselves assured of their unworthiness, he promised their hands to two who could perform a set prophecy task. The suitors were led in front of two tables on each of which stood three boxes in a row, and urged to say which boxes contained a gem and which were empty. Yet, as many as would try it, it appeared impossible to perform the task. Indeed, after they both had made their prophecies, each of the suitors was urged by the father to open two boxes that he had named as both empty or as both not empty: it always proved to be that one contained a gem and the other did not, and in fact the gem lay now in the first, now in the second of the opened boxes, and yet, every time the suitors both opened the first box, or both the second, or both the third, whatever one of them found (or failed to find) in one box, the other would also find (or fail to find) in the corresponding box (which showed that the gems needs must have been contained in the boxes in the first place). But how should it be possible, out of three boxes, to name no two as empty or as not empty?

Thus indeed the daughters would have remained unmarried until their father's death, had they not urged two bright young students to attempt the task, with whom they were secretly in love, and whose names, let it be known, were Sandu and Daniel. Now, Sandu and Daniel were not renowned at the time as having any particular gift for prophecy, but they were very ingenious and hard-working, and Daniel's uncle was a prophet of considerable standing and Daniel hoped to have inherited some of his gift. The two friends were also both desperately in love, so they picked up their courage and sought an audience with the seer. When he heard Sandu and Daniel asking him for his daughters' hands, the seer smiled and (judging they had not the slightest chance of succeeding in the task) declared that not only were they welcome to come upon the morrow to test their skills, but he would give them not one, nor two, but one hundred chances, if they would care to try. And so the father was awake all night setting up two hundred little tables and six hundred boxes and placing gems from his collection in at least two hundred of them (for he had won many prizes in prophecy competitions over the years). And the next morning Sandu and Daniel duly presented themselves to the seer's home and embarked on making prophecies. As soon as they had both performed a set of prophecies, at the father's urging each of them opened two boxes that he had named as both empty or as both not empty, and, lo and behold, it always proved to be that one contained a gem and the other did not, and the gem lay now in the first, now in the second of the opened boxes, and yet, every time they had both opened the first box, or both the second, or both the third, whatever one of them found (or failed to find) in a box, the other would also find (or fail to find) in the corresponding box, just as had happened to every pair of suitors before them (and as the two daughters had long grown accustomed to). And so they laboured all morning (for prophesising is a strenuous task, especially if you are still a student), becoming ever more disconsolate at each unsuccessful attempt. At the end of fifty trials, the father called a break, and kindly offered Sandu and Daniel some refreshment and some cucumber sandwiches, of which they partook, grateful of a rest and of the opportunity to exchange some precious words with their beloved ones. When the hour was over and the final fifty trials were to begin, the younger of the two daughters went up to her father and spoke to him thus: ‘Father, Sandu and Daniel realise that you are a far better prophet than they are, and that if you choose the boxes that are to be opened, then not in a single trial will their prophecies stand. My sister and I therefore humbly beseech you (for we are fond of them, even though you must be laughing at their youth and inexperience) to give them one small chance of success. If you will, demand they be successful in all fifty trials that still lie before them, but please let them choose themselves which boxes shall be opened and prophesise which shall be found empty or full. If they both fail, they will lay down any claim to our hands forever. But if at least one of them succeeds (for neither of us could ever be happy if the other is not), oh please let your loving daughters be married to them, and we shall forever and a day be grateful to you, dear Father'. Now, well knowing how tiny the chance was of either Sandu or Daniel successfully predicting fifty trials at equal odds (unless either had a very great gift for prophecy, indeed), but equally moved by this eloquence (because the younger daughter who had thus spoken had her own subtle gift of winding her father around her little finger), the father consented to this alteration in the remaining trials, as long of course as Sandu and Daniel still made their prophecies before seeing each other's results. And so, Sandu and Daniel embarked again on making their prophecies for the rest of the day. Upon each trial, first Daniel would announce which two boxes he would open, prophesise whether the first would be full or empty, and accordingly prophesise that the second would be empty or full, respectively. Then Sandu would choose his first box and both prophesise and verify whether this be full or empty, then he would choose his second box and accordingly both prophesise and verify that it would be empty or full, respectively. Finally Daniel would open his two chosen boxes and verify his own prophecies. Upon the first trial, both Sandu's and Daniel's prophecies proved to be successful, and so they did upon the second trial, and upon the third. Upon the fourth trial, however, Sandu's first prophecy was falsified, and in the subsequent trials it became apparent that of each pair of Sandu's prophecies, the first one met now with success, now with failure, as if he had no gift for prophecy at all (the second of course was always successful). But the father had already turned pale by the second trial, realising he had been outsmarted. And, indeed, Daniel, whether his family gift had found his way to him at last, or through some other artifice, and to the increasing surprise and delight of the numerous bystanders (for by this time the father's servants and neighbours had started gathering around), kept meeting with success, trial upon trial upon trial. Just before the hundredth and last trial, the father interjected, protesting weakly that he had not meant Sandu to announce his second prophecy only after having already verified the first, to which now the elder of his daughters replied: What difference did that make to Sandu's chances of success (for it made no difference to his first prophecy, and it made a difference to the second only when he had failed in the first one already)? And so, the father grumbling let the final trial proceed, and Daniel triumphantly extracted a sparkling emerald (matching one that Sandu had just extracted) from the last of his boxes, which he had indeed prophesised to be full. The four young people were married the very next day, and henceforth and for the rest of their lives both Daniel and Sandu enjoyed a reputation as formidable prophets. Meanwhile, the father consoled himself in the knowledge of having brought up two very clever daughters, indeed.

Main result:
Our bipartite 3-box scenario is to be contrasted with Liang, Spekkens and Wiseman's, who consider only one box opened on each side, with perfect correlations for matching boxes, perfect anticorrelations for different ones (yielding a realisation of a PR box).

Note that if Alice opens A and Bob opens C (getting the opposite result), then by no-signalling if they were after all to both open B, the anti-correlations on either side would be preserved, enforcing *anti*-correlation between B and B.

In our scenario, if we measure B on both sides, we get perfect correlations irrespective of what else we measure on the two sides.

QM analogy in their scenario: measuring spins in 3 directions on two spin-1/2 systems in the singlet state (pairwise compatible *in that state*): measurements disentangle the state.

QM analogy in our scenario: measuring 1-dimensional projections on two spin-1 systems in the singlet state (as in Kochen and Specker 1967): perfect correlations irrespective of other measurements.

With this assumption, we can indeed implement a protocol for signalling:

- Daniel announces he will open A and B
- Sandu opens C and gets e.g. -1
- He now can choose to open A or B, in *either* case getting $+1$.
- Thereby he can choose which of *Daniel's* A or B will give $+1$.

Somewhat more formally, assume we have some suitable propositional structure allowing for a suitable tensor product (tricky: we shall return to this!).

Assume further:

(a) existence of maximally entangled states

(b) no-signalling

(c) ‘robustness of correlations under compatible measurements’

Remarks:

(a) means we assume we can take the composition of a system with itself, and define a state such that for all propositions A, the product $A \otimes A$ exhibits perfect correlations.

(b) has its usual meaning.

(c) we shall return to.

We can now prove Specker's principle as follows:

- Assume we have a system with a set of n pairwise orthogonal propositions that are not jointly orthogonal.
- Assume all its proper subsets are jointly orthogonal (otherwise select an appropriate subset).
- Take two copies of the system in a maximally entangled state (call them Daniel's and Sandu's).

- Daniel now chooses two questions A and B.
- Sandu tests all other $n-2$ questions on his subsystem; at most one tests positive. Take one at random if they all tested negatively, otherwise take the one that tested positively. Call this C.
- The questions A, B, C on the two sides are now isomorphic to the boxes in the fable, and the rest of the protocol for signalling is as above.

Discussion:

(1) The mechanism in the fable is neither ‘instantaneous collapse’ nor ‘action at a distance’, but retrocausal.

The father wants to see if the suitors are better prophets than himself, and predicts *which four boxes will be opened* each time, filling them accordingly after flipping a coin. If he is right, the suitors do not have a chance in the original game, but also have a guaranteed winning strategy in the revised game.

In the original one-suitor fable, retrocausation remains hidden. Here, it becomes manifest and exploitable.

(2) Also in our scenario we get a realisation of PR boxes.

Indeed, let Daniel choose between AB or CA (a or a'), and let Sandu choose between AB or BC (b or b').

Interpret Daniel's a and a' as (A in the context B) and (C in the context A), and Sandu's b and b' as (A in the context B) and (B in the context C).

Then $< ab >= 1, < ab' >=< a'b >=< a'b' >= -1$. But each of Daniel's and Sandu's measurements also has an interpretation as (B in the context A), (A in the context C), etc. By choosing different interpretations (16 combinations) we get 2 realisations each of all 8 PR boxes.

(3) What about the 'robustness' assumption (c)?

Stated in greater generality than needed, (c) means: if A' is compatible with A and B' is compatible with B, then $A' \otimes B'$ is compatible with $A \otimes B$.

But if we assume we have a *state* on the composite, and if we assume that $A \otimes B$ is indeed a proposition of the composite, then the probabilities assigned by the state to $A \otimes B$ must be *independent* of how $A \otimes B$ is measured. Thus (c) is already required if we have a 'suitable' notion of tensor product.

Of course, it could be that a measurement of A in the context A' and one of B in the context B' do not constitute a measurement of the product $A \otimes B$, but of a *new* proposition that needs to be distinguished from it.

In that case, assumption (c) becomes the requirement that if measuring A' is an allowable context for measuring A, and measuring B' is an allowable context for measuring B, then measuring A' and measuring B' is an allowable context for measuring $A \otimes B$.

Possibly an obvious requirement, and at the very least physically transparent!

Note that one of the main lessons of the quantum logic literature up to the 1980s was that it is difficult to define a notion of tensor product for propositional structures.

The above can be see in part as a belated contribution to that debate.

(4) It is known that Specker's principle gets closer to QM than the combination of Bell non-locality and no-signalling. But our result may perhaps be strengthened further in two directions:

(4a) Requiring joint orthogonality of pairwise orthogonal propositions is a weakened form of the original Specker principle used as an axiom by Kochen and Specker. The original principle was: *pairwise compatible propositions are jointly compatible.*

Can one adapt the argument above to prove also this *strong Specker principle*?

(4b) Violation of the (weak) Specker principle was used above to construct a non-quantum Kochen-Specker witness, which was then combined with perfect correlations.

In QM, the combination of KS witnesses with perfect correlations yields ‘algebraic proofs of nonlocality’ (*Stairs’ theorem*).

Can one generalise the argument to rule out other non-quantum examples of Kochen-Specker witnesses (Stairs nonlocality and no-signalling as axioms for QM)?