Geometric probability theory in contextual probabilistic theories

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Outline

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- We look for group actions and invariant states in (possibly) contextual probabilistic theories.
- These notions are important in formulating symmetries, constrains in the MaxEnt principle and physical principles in general.
- We look for a formal framework based on measure theory: after all, probabilities can be considered as measures over suitable algebraic structures.
- We explore the possibility of developing a non-commutative version of *geometric probability theory*.
- Joint work with Cesar Massri (CONICET) and Angel Plastino (CONICET).

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Classical probability

Probability measures

$$\mu : \Sigma \to [0, 1] \tag{1}$$

such that:

1. $\mu(\emptyset) = 0$
2. $\mu(A^c) = 1 - \mu(A)$
3. For each family of pairwise disjoint sets $\{A_i\}_{i \in I}$

$$\mu(\bigcup_{i \in I} A_i) = \sum_i \mu(A_i)$$

Classical case

$\sigma : \Gamma \longrightarrow [0; 1]$, such that $\int_\Gamma \sigma(p, q) d^3p d^3q = 1$

Convex set of quantum states

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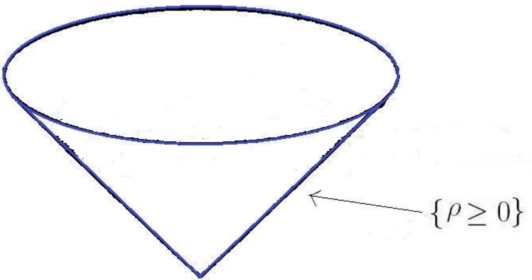
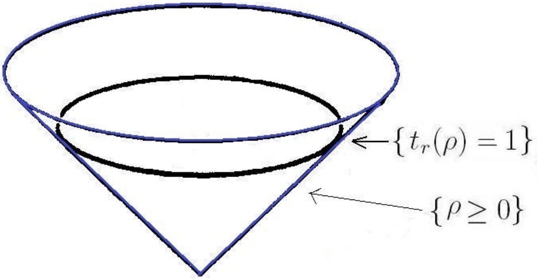


Figure: Geometric representation.

Convex set of quantum states



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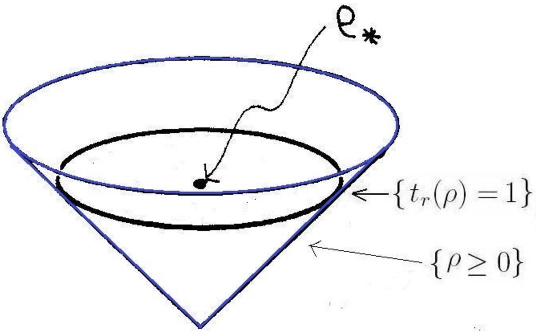

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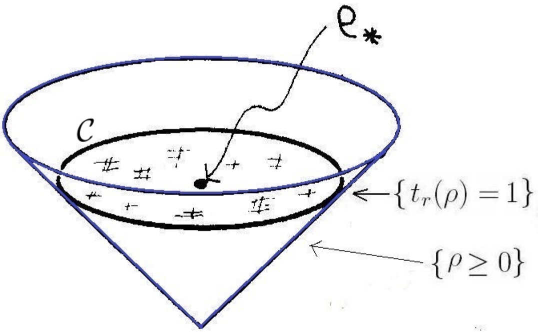

Figure: Geometric representation.

Quantum probability (Born's rule)

Probability measures

$$s : \mathcal{L}_{v\mathcal{N}} \longrightarrow [0; 1] \tag{2}$$

such that:

1. $s(\mathbf{0}) = 0$ ($\mathbf{0}$ null subspace).
2. $s(P^{\perp}) = \mathbf{1} - s(P)$
3. for each family of pairwise orthogonal projections (P_j), $s(\sum_j P_j) = \sum_j s(P_j)$

Gleason's theorem

$$s_{\rho}(P) = tr(\rho P) \tag{3}$$

[F. Holik, A. Plastino and M. Sáenz, *Annals Of Physics*, Volume **340**, Issue **1**, 293-310, (2014)]

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Gleason's theorem

Gleason's theorem assures that there exists a density matrix for each probability measure as defined above ($dim(\mathcal{H}) \geq 3$).

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More general theories

- In a series of papers Murray and von Neumann searched for algebras more general than $\mathcal{B}(\mathcal{H})$.
- The new algebras are known today as von Neumann algebras, and their elementary components can be classified as Type I, Type II and Type III factors.
- It can be shown that, the projective elements of a factor form an orthomodular lattice. Classical models can be described as commutative algebras.
- The models of standard quantum mechanics can be described by using Type I factors (Type I_n for finite dimensional Hilbert spaces and Type I_∞ for infinite dimensional models). These are algebras isomorphic to the set of bounded operators on a Hilbert space.

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More general theories

- Further work revealed that a rigorous approach to the study of quantum systems with infinite degrees of freedom needed the use of more general von Neumann algebras, as is the case in the axiomatic formulation of relativistic quantum mechanics. A similar situation holds in algebraic quantum statistical mechanics.
- In these models, States are described as complex functionals satisfying certain normalization conditions, and when restricted to the projective elements of the algebras, define measures over lattices which are not the same to those of standard quantum mechanics.

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von Neumann algebras

- Canonical example of a von Neumann algebra: $\mathcal{B}(H)$
- The easiest way to define a von Neumann algebra regards it as a $*$-subalgebra $\mathcal{W}$ satisfying $\mathcal{W}'' = \mathcal{W}$, where given $S \subseteq \mathcal{B}(H)$, S' is defined as

$$S' = \{A \in \mathcal{B}(H) \mid AB - BA = 0 \, \forall B \in S\}$$

- The collection of orthogonal projections of a von Neumann algebra $\mathcal{W}$ is an orthomodular lattice $\mathcal{P}(\mathcal{W})$.

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States

- A *state* $\nu : \mathcal{W} \longrightarrow \mathbb{C}$ is defined as a continuous positive linear functional such that $\nu(\mathbf{I}) = 1$.
- Positivity means that $\nu(A^*A) \geq 0$ for all $A \in \mathcal{W}$ or, equivalently, that $\nu(A) \geq 0$ for all $A \geq 0$
- *Normal states* can be defined as those states satisfying the condition $\nu(\sup_\alpha(a_\alpha)) = \sup_\alpha \nu(a_\alpha)$ for any uniformly bounded increasing net a_α of positive elements of $\mathcal{W}$ (equivalently, states satisfying $\nu(\sum_{i\in I} E_i) = \sum_{i\in I} \nu(E_i)$ for any countable and pairwise orthogonal family of events $\{E_i\}_{i\in I}$).

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Why orthomular?

Thus, normal states of physical theories define probabilities on orthomodular lattices satisfying the following properties:
Let $\mathcal{L}$ be an orthomodular lattice. Then, define

$$s : \mathcal{L} \rightarrow [0;1],$$

($\mathcal{L}$ standing for the lattice of all events) such that:

$$s(\mathbf{0}) = 0. \tag{4}$$

$$s(E^{\perp}) = 1 - s(E),$$

and, for a denumerable and pairwise orthogonal family of events E_j

$$s(\sum_j E_j) = \sum_j s(E_j).$$

where $\mathcal{L}$ is a general orthomodular lattice (with $\mathcal{L} = \Sigma$ and $\mathcal{L} = \mathcal{P}(\mathcal{H})$ for the Kolmogorovian and quantum cases respectively).

Maximal Boolean subalgebras

Maximal Boolean subalgebras

- An orthomodular lattice $\mathcal{L}$ can be described as a pasting of Boolean algebras:

$$\mathcal{L} = \bigvee_{\mathcal{B} \in \mathfrak{B}} \mathcal{B}$$

(where $\mathfrak{B}$ is the set of maximal Boolean algebras of $\mathcal{L}$).

- A state s of $\mathcal{L}$ defines a classical probability on each classical Boolean subalgebra $\mathcal{B}$. In other words: $s_{\mathcal{B}}(\ldots) := s|\mathcal{B}(\ldots)$ is a Kolmogorovian measure over $\mathcal{B}$.

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But then...

- One can think about much more general theories.
- In fact, more general non-Kolmogorovian structures have been found associated to problems in biology, cognition and computer science.
- This has direct implications for information theory: F. Holik, G. M. Bosyk and G. Bellomo, "Quantum Information as a Non-Kolmogorovian Generalization of Shannon's Theory", *Entropy* 2015, **17** (11), 7349-7373.
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Axioms of geometric probability theory

Axiom1

$\mu(\emptyset) = 0$

Axiom2

If A and B are measurable sets: $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ which (for Boolean algebras) is equivalent to:

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{5}$$

for disjoint A and B.

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The following axiom reflects the action of a group that leaves the measure invariant:

Axiom3

The volume of a set A does not depends on the position of A; in other words, if A can be rigidly transformed in B, then, the volumes (measures) of B and A are equal.

Axioma

Given a parallelotope P, with orthogonal sides $x_1, \ldots, x_n$, we impose the normalization condition: $\mu(P) = x_1 x_2 \cdots x_n$

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Symmetric polynomials

$$e_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n \tag{6a}$$

$$e_2(x_1, x_2, \ldots, x_n) = x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \tag{6b}$$

$$e_{n-1}(x_1, x_2, \ldots, x_n) = \quad x_2 x_3 \cdots x_n + x_1 x_3 x_4 \ldots x_n + \ldots + x_1 x_2 \cdots x_{n-1} \tag{6c}$$

$$e_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n \tag{6d}$$

Each one of these polynomials gives a different invariant measure.

Generalization

Consider a function:

$$s : \mathcal{L} \to [0;1], \tag{7}$$

Axiom1

$s(\mathbf{0}) = 0$

Axiom2

For an orthogonal and denumerable family E_j, we have

$$s(\sum_j E_j) = \sum_j s(E_j)$$

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Generalization

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There exists a group of automorphisms $\mathcal{F}$ such that for all $g \in \mathcal{F}$ amd all $E \in \mathcal{L}$

$$s(g \cdot E) = s(E)$$

Axiom4

The normalization condition has the form

$$e_i(s(E_1), s(E_2), \ldots) = 0$$

where $\{E_j\}_J \subseteq \mathcal{L}$ is a subset of events.

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Generalization

Maximization process

- These axioms determine a convex set $\mathcal{C}$ in a univocal way and axiom 3 determines a variety $\mathcal{M}$.
- The set of states of a concrete physical system can be described as $\mathcal{C} \cap \mathcal{M}$.
- We compute the measurement entropy on this set:

$$H_E(s) := -\sum_{x \in E} s(x) \ln(s(x))$$

$$H(s) := \inf_{E \in \mathcal{L}} H_E(s)$$

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Conditions

Consider the conditions:

$$\begin{aligned} \langle R_1 \rangle &= r_1 \\ \langle R_2 \rangle &= r_2 \\ &\vdots \\ \langle R_n \rangle &= r_n, \end{aligned} \tag{8}$$

We want to determine the least biased probability distribution satisfying these constrains.

MaxEnt tell us that:

$$\rho_{max-ent} = \exp^{-\lambda_0 \mathbf{1} - \lambda_1 R_1 - \cdots - \lambda_n R_n}, \tag{9}$$

where the λs are Lagrange multipliers satisfying:

$$r_i = -\frac{\partial}{\partial \lambda_i} \ln Z, \tag{10}$$

and

$$Z(\lambda_1 \cdots \lambda_n) = \mathrm{tr}[\exp^{-\lambda_1 R_1 - \cdots - \lambda_n R_n}], \tag{11}$$

The normalization condition reads:

$$\lambda_0 = \ln Z. \tag{12}$$

Geometric description

Given an effect E, let us consider the set of states:

$$C_{(E,\lambda)} := \{\rho \in \mathcal{C} \,|\, \mathrm{tr}(\rho E) = \lambda, \;\; \lambda \in [0,1]\}. \tag{13}$$

It is a convex set nd there exists S (a real subspace in $\mathcal{A}$) such that:

$$C_{(E,\lambda)} = S \cap \mathcal{C}, \tag{14}$$

In a generalized model

In general, an equation of the form:

$$\langle R \rangle = r, \tag{15}$$

Geometric characterization

...can be expressed as subspace intersected with the convex set of states: $S(R,r) \cap \Omega$.

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In generalized models:

$$C_{max-ent} := \bigcap_i C_{R_i} = \bigwedge_i C_{R_i}. \tag{16}$$

Given a series of conditions represented by convex sets C_i, one should maximize entropy in $C_{max-ent} = \bigwedge_i C_i$.
[F. Holik and A. Plastino, *Journal Of Mathematical Physics* **53**, 073301 (2012)]

Functor

Let $\mathcal{L}$ be an orthocomplemented lattice. Then, there exists an abelian group $M = M(\mathcal{L})$ such that the functor $\mathcal{M}(\mathcal{L}; -)$ satisfies,

$$\mathcal{M}(\mathcal{L}; -) = \mathrm{Hom}_{\mathbb{Z}}(M, -)$$

This means that a measure in $\mathcal{L}$ valued in A is equivalent to a $\mathbb{Z}$-linear map from $M(\mathcal{L})$ to A.

Measures factorize

Let $\mathcal{L}$ be an orthocomplemented lattice and assume that a group G acts by automorphism. Let A be an abelian group where G acts trivially.

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\forall \nu"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\ M(\mathcal{L}) \arrow[ur, dashed, "\exists ! \overline{\nu}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "\forall \nu'"] \arrow[d, "\pi_G"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\ M(\mathcal{L})_G \arrow[ur, dashed, "\exists ! \overline{\nu'}"'] & \end{tikzcd}$$

where ν (resp. ν') is a measure (resp. invariant measure) and $\overline{\nu}$, $\overline{\nu'}$ are linear maps. The commutativity means that $\nu = \overline{\nu}\pi$, $\nu' = \overline{\nu'}\pi_G$.

[Non-Boolean Groemer's integral theorem] Let $\mathcal{L}$ be an orthocomplemented lattice where a group G acts. Let B be an orthogonal generating set for the action of G. Then, invariant measures on $\mathcal{L}$ are in bijection with $N_G(B)$-invariant functions on B, ν, such that

$$\nu(b_1 \vee b_2) = \nu(b_1) + \nu(b_2), \quad \forall b_1, b_2, b_1 \vee b_2 \in B, \quad b_1 \perp b_2.$$

[C. Massri, F. Holik and A. Plastino, "States in generalized probabilistic models: an approach based in algebraic geometry", arXiv:1705.03045v1 [quant-ph] (2017).]

Conclusions

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- We study invariant measures in a general framework that includes many contextual theories of interest.
- We approach the problem from the perspective of measure theory. More precisely, we present a non-commutative version of geometric probability theory. Formulating the problem in terms of invariant measures allows link states and group actions in a natural way.
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Some references

References

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