A complete characterisation of All-versus-Nothing arguments on stabiliser states

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All-vs-Nothing (AvN) Arguments

A type of contextuality proof which rests on the derivation of an inconsistent system of parity equations.

Originally introduced by Mermin

c.f. *A Simple Unified Form For the Major No-Hidden-Variables Theorems* (PRL 1990).

It has been used to prove **strongly contextual behaviour** in the empirical model arising from applying Pauli measurements on the GHZ state.

Many other studies have appeared subsequently, with a variety of instances of AvN arguments.

Recent work by Abramsky et al. has provided a **general, formal definition** of what an AvN argument actually is

cf. *Contextuality, Cohomology and Paradox* (Proceedings of CSL 2015)

Motivation: Characterise the quantum states which give rise to maximal degrees of non-locality/contextuality

Present work: A partial answer to this question for the case of **stabiliser states**.

Mermin's original All-vs-Nothing argument

The **Pauli matrices**:

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dichotomic observables with eigenvalues ± 1. They satisfy the following equations:

$$\begin{gathered} X^2 = Y^2 = Z^2 = I \\ XY = iZ, \quad YZ = iX, \quad ZX = iY, \\ YX = -iZ, \quad ZY = -iX, \quad XZ = -iY. \end{gathered}$$

Consider a tripartite measurement scenario where each party $i = 1, 2, 3$ can perform Pauli measurements X or Y on the GHZ state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

with outcomes in $\mathbb{Z}_2$, where we have relabeled $(+1, -1, \cdot) \cong (0, 1, \oplus)$.

Mermin's All-vs-Nothing Argument

A portion of the support of the empirical model:

1	2	3	000	001	010	011	100	101	110	111
...	...	...	...	...	...	...	...	...	...	...
X_1	X_2	X_3	1	0	0	1	0	1	1	0
X_1	Y_2	Y_3	0	1	1	0	1	0	0	1
Y_1	X_2	Y_3	0	1	1	0	1	0	0	1
Y_1	Y_2	X_3	0	1	1	0	1	0	0	1
...	...	...	...	...	...	...	...	...	...	...

We can see that any non-contextual assignment of outcomes to each variable must satisfy the following equations in $\mathbb{Z}_2$:

$$\bar{X}_1 \oplus \bar{X}_2 \oplus \bar{X}_3 = 0 \qquad \bar{Y}_1 \oplus \bar{X}_2 \oplus \bar{Y}_3 = 1$$
$$\bar{X}_1 \oplus \bar{Y}_2 \oplus \bar{Y}_3 = 1 \qquad \bar{Y}_1 \oplus \bar{Y}_2 \oplus \bar{X}_3 = 1.$$

When summed, the equations yield $0 = 1$, showing that it is impossible to find a global assignment $g : \{X_{1,2}, Y_{1,2}\} \to \mathbb{Z}_2$ compatible with the event deemed possible by the empirical model. This means that the model is **strongly contextual**.

General setting

It is possible to generalise Mermin's argument to any **empirical model** over a measurement scenario $(X, \mathcal{M})$ with dichotomic measurements.

Let $e = \{e_U\}_{U \in \mathcal{M}}$ be an empirical model. We can associate to it an **XOR theory** $\mathbb{T}_\oplus(e)$, defined as follows:

For each set $U \in \mathcal{M}$ of jointly performable measurements, the theory $\mathbb{T}_\oplus(e)$ will contain the assertion

$$\bigoplus_{x \in U} \overline{x} = 0$$

when the support of e_U only contains joint outcomes of even parity (i.e. with an even number of 1s), and

$$\bigoplus_{x \in U} \overline{x} = 1$$

when the support of e_U only contains joint outcomes of odd parity (here, $\overline{x} \in \mathbb{Z}_2$ denotes a variable for the outcome of measurement x).

We say that the model e is **AvN** if $\mathbb{T}_\oplus(e)$ is inconsistent.

Proposition

If an empirical model is AvN, then it is strongly contextual.

The Stabiliser World

Stabiliser quantum mechanics is a natural setting for general AvN arguments.

The Pauli *n*-group P_n: sequences $\alpha(P_i)_{i=1}^n$ of n Pauli operators (P_i in $\{X, Y, Z, I\}$), with a global phase $\alpha \in \{\pm 1, \pm i\}$. The group P_n acts on the Hilbert space $H_n := (\mathbb{C}^2)^{\otimes n}$ of n-qubit states as follows:

$$\alpha(P_i)_{i=1}^n \bullet |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle := \alpha P_1|\psi_1\rangle \otimes \cdots \otimes P_n|\psi_n\rangle$$

Given a subgroup S of P_n, we denote by

$$V_S := \{|\psi\rangle \mid P \bullet |\psi\rangle = |\psi\rangle \ \forall P \in S\}$$

the linear subspace of H_n of states stabilised by S. Note that subgroups stabilising non-trivial subspaces must be **abelian**, and only contain elements with **global phase** ± 1. Such subgroups are called **stabiliser subgroups**

Stabiliser subgoups of P_n give rise to empirical models obtained by applying measurements in S on a state in V_S.

There is an important relation between S and V_S:

$$\text{rank } S = k \quad \Leftrightarrow \quad \dim V_S = 2^{n-k}$$

If S is a **maximal stabiliser subgroup** (i.e. rank $S = n$), then it stabilises a unique state (up to phase). Such a state is called a **stabiliser state**.

Subgroups of P_n induce XOR theories

Let $P := \alpha(P_i)_{i=1}^n \in \mathrm{P}_n$, and $|\psi\rangle$ be a state stabilised by P (i.e. $\alpha = (-1)^a$, for some $a \in \mathbb{Z}_2$). Then

$$\alpha(P_1 \otimes \cdots \otimes P_n)|\psi\rangle = |\psi\rangle$$

and so $|\psi\rangle$ is an α-eigenvector of $P_1 \otimes \cdots \otimes P_n$. Thus the expected value satisfies

$$\langle\psi|P_1 \otimes \cdots \otimes P_n|\psi\rangle = \alpha = (-1)^a$$

Hence, an assignment of outcomes to $P_1, \ldots, P_n$ consistent with the model has to satisfy the following assertion:

$$\phi_P := \left(\bigoplus_{\substack{i\in\{1,\ldots,n\} \\ P_i \neq I}} \bar{P}_i \;=\; a \right)$$

This means that to each subgroup $S \leq \mathrm{P}_n$ we can associate an XOR theory $\mathbb{T}_\oplus(S)$

$$\mathbb{T}_\oplus(S) := \{\phi_P \mid P \in S\},$$

We say that S is **AvN** if $\mathbb{T}_\oplus(S)$ is inconsistent.

Proposition

If S is AvN, then any empirical model associated to it is strongly contextual.

Galois connection

It is possible to formalise the connection between subgroups of P_n and subspaces of H_n using the theory of **Galois connections**.

In particular, the assignments

$$\begin{array}{ccc} \mathcal{P}(\mathsf{P}_n) & \longleftrightarrow & \mathcal{P}(H_n) \\ S & \longmapsto & S^\perp := V_S \\ V^\perp := \bigcap_{v \in V} (\mathsf{P}_n)_v & \longleftarrow & V \end{array} \qquad (1)$$

constitute an antitone Galois connection (i.e. $S \subseteq S^{\perp\perp}$ and $V \subseteq V^{\perp\perp}$). The **closed sets** $S^{\perp\perp}$ and $V^{\perp\perp}$ of the connection $\mathcal{P}(\mathsf{P}_n) \leftrightarrow \mathcal{P}(H_n)$ are subgroups of P_n and subspaces of H_n respectively. If we restrict (1) to closed sets, we obtain a new Galois connection

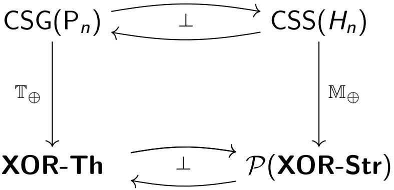
$$\mathsf{CSG}(\mathsf{P}_n) \longleftrightarrow \mathsf{CSS}(H_n).$$

which closely resembles the one between **syntax and semantics** in logic:

$$\begin{array}{ccc} \mathcal{L}\textbf{-Theories} & \longleftrightarrow & \mathcal{P}(\mathcal{L}\textbf{-Structures}) \\ \Gamma & \longmapsto & \Gamma^\perp := \{\mathcal{M} \mid \forall \varphi \in \Gamma . \mathcal{M} \models \varphi\} \\ M^\perp := \{\varphi \mid \forall \mathcal{M} \in M . \mathcal{M} \models \varphi\} & \longleftarrow & M, \end{array}$$

AvN arguments and logical paradoxes

More precisely, we can establish a formal relation between the two connections in the case of XOR-theories:



The function $\mathbb{T}_\oplus$ maps S to its XOR-theory, while $\mathbb{M}_\oplus$ maps a stabiliser state to the set of global assignments compatible with the empirical model defined by it.

In categorical terms, these two maps constitute a **monomorphism of adjunctions**, and allow us to formally describe how AvN arguments can be seen as logical paradoxes: to a quantum realisable strongly contextual model corresponds an inconsistent logical theory.

How can we characterise the AvN subgroups of P_n?

AvN triples

We introduce the notion of **AvN triple**:

Definition

An **AvN triple** in P_n is a triple $\langle e, f, g \rangle$ of elements of P_n with global phases ± 1 that satisfy the following conditions:

1. For each $i = 1, \ldots, n$, at least two of e_i, f_i, g_i are equal.
2. The number of i such that $e_i = f_i \neq g_i$, all distinct from I, is odd.
3. The number of i such that $e_i \neq f_i = g_i$, all distinct from I, is odd.
4. The number of i such that $e_i = g_i \neq f_i$, all distinct from I, is odd.

Example

The original Mermin argument is based on the following AvN triple:

$$\begin{array}{ccc} X_1 & Y_2 & Y_3 \\ Y_1 & X_2 & Y_3 \\ Y_1 & Y_2 & X_3 \end{array}$$

AvN triples

Previous work (*Contextuality, Cohomology and Paradox*, CSL 2015) has proved that AvN triples provide a sufficient condition for AvN contextuality:

Theorem

If a subgroup S contains an AvN triple, than it is AvN.

Remarkably, every AvN argument which has appeared in the literature can be seen to come down to exhibiting an AvN triple:

Conjecture (AvN triple conjecture)

A subgoup S is AvN if and only if it contains an AvN triple.

We will prove the AvN triple conjecture in the case of **maximal stabiliser subgroups** (i.e. for **stabiliser states**).

Graph states

Graph states are special types of multi-qubit states that can be represented by a graph.

Let $G = (V, E)$ be an undirected graph. For each $u \in V$, consider the element $g^u = (g^u_v)_{v \in V} \in \mathsf{P}_{|V|}$ with global phase $+1$ and components

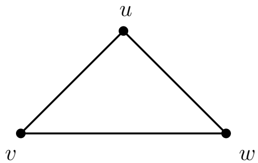
$$g^u_v = \begin{cases} X & \text{if } v = u \\ Z & \text{if } v \in \mathcal{N}(u) \\ I & \text{otherwise} \end{cases}$$

The **graph state** $|G\rangle$ associated to G is the unique state stabilised by the subgroup generated by these elements:

$$S_G = \langle \{ g^u \mid u \in V \} \rangle$$

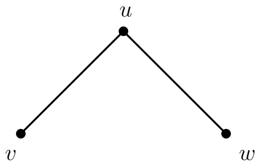
Graph states

Two simple examples:



G

$$S_G = \left\langle \begin{array}{cccccc} g^u & : & X_u & Z_v & Z_w \\ g^v & : & Z_u & X_v & Z_w \\ g^w & : & Z_u & Z_v & X_w \end{array} \right\rangle$$



G'

$$S_{G'} = \left\langle \begin{array}{cccccc} g^u & : & X_u & Z_v & Z_w \\ g^v & : & Z_u & X_v & I_w \\ g^w & : & Z_u & I_v & X_w \end{array} \right\rangle$$

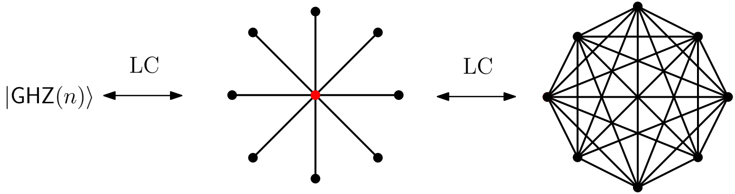
Graph states

A key result, due to Schlingemann:

Theorem

Any stabiliser state $|S\rangle$ is LC-equivalent to some graph state $|G\rangle$, i.e. $|S\rangle = U|G\rangle$ for some local Clifford unitary U.

An important example:



Remark: Both contextual properties and AvN triples are preserved under Local Clifford operations. Hence, for our purposes, given a stabiliser state we can assume w.l.o.g. that it is a graph state.

AvN triple theorem

Theorem (AvN triple theorem)

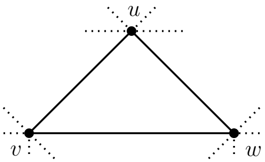
A maximal subgroup S of P_n is AvN if and only if it contains an AvN triple. The AvN argument can be reduced to one concerning only three qubits. The state induced by the subgraph for these three qubits is LC-equivalent to a tripartite GHZ state.

Sketch of the proof: Let $|S\rangle$ denote the stabiliser state associated to S. W.l.o.g. $|S\rangle = |G\rangle$ for some graph $G = (V, E)$, $|V| = n$.

Case 1: The maximal degree of G is strictly smaller than 2. Then $|G\rangle$ is a tensor product of 1-qubit and 2-qubit states, which is not enough to obtain strong contextuality.

AvN triple theorem

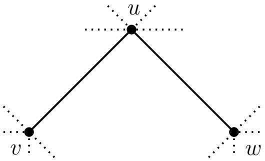
Case 2: There is a vertex u with degree ≥ 2. Then we have two cases:



G

$$S_G = \left\langle \begin{array}{ccccc|c} g^u & : & X_u & Z_v & Z_w & [I \text{ or } Z] \\ g^v & : & Z_u & X_v & Z_w & [I \text{ or } Z] \\ g^w & : & Z_u & Z_v & X_w & [I \text{ or } Z] \end{array} \right\rangle$$

Then $\langle g^u, g^v, g^w \rangle$ is an AvN triple



G

$$S_G = \left\langle \begin{array}{ccccc|c} g^u & : & X_u & Z_v & Z_w & [I \text{ or } Z] \\ g^v & : & Z_u & X_v & I_w & [I \text{ or } Z] \\ g^w & : & Z_u & I_v & X_w & [I \text{ or } Z] \end{array} \right\rangle$$

Then $\langle g^u, g^u g^v, g^u g^w \rangle$ is an AvN triple. The only relevant qubits in the AvN argument are u, v, w. The states induced by the two possible subgraphs for u, v, w are both LC-equivalent to GHZ.

Applications of the AvN triple theorem

The combinatorial properties of AvN triples allow us to better understand AvN arguments from a computational perspective.

Proposition

Let $n \geq 3$. The number of AvN triples in P_n is given by

$$8 \cdot \left[\sum_{k=1}^{\frac{1}{2}(n+[n])-1} \binom{n}{2k+1} \binom{k+1}{k-1} \cdot 6^{2k+1} \cdot 22^{n-2k-1} \right],$$

where $[n] \in \mathbb{Z}_2$ denotes the parity of n.

$n = 3 : 1'728$

$n = 4 : 152'064$

$n = 5 : 8'550'144$

Applications of the AvN triple theorem

The check vector representation of P_n: Given an element $P := \alpha(P_i)_{i=1}^n \in \mathsf{P}_n$, its check vector $r(P)$ is a $2n$-vector

$$r(P) = (x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n) \in \mathbb{Z}_2^{2n}$$

whose entries are defined as follows

$$(x_i, z_i) = \begin{cases} (0,0) & \text{if } P_i = I \\ (1,0) & \text{if } P_i = X \\ (1,1) & \text{if } P_i = Y \\ (0,1) & \text{if } P_i = Z. \end{cases}$$

Every check vector $r(P)$ completely determines P up to phase (i.e. $r(P) = r(\alpha P)$ for all $\alpha \in \{\pm 1, \pm i\}$). We can take advantage of this representation to develop a computational method capable of generating all the possible AvN triples in P_n and, therefore, all the AvN arguments for n-qubit stabiliser states.