On the non-existence of two-valued lattice homomorphisms of quantum logic

Minoru Koga

Nagoya University, Graudate School of Informatics

koga.minoru@d.mbox.nagoya-u.ac.jp

NAGOYA UNIVERSITY

Abstract

We show that there exists no two-valued lattice homomorphism from the lattice $\mathscr{L}(\mathscr{H})$ of all closed linear subspaces of a Hilbert space with $\dim(\mathscr{H}) \geq 2$ by using the notion of prime filters.

Introduction

The lattice $\mathscr{L}(\mathscr{H})$ of all closed linear subspaces of a Hilbert space $\mathscr{H}$ can be seen as the set of experimental propositions of a quantum system, and its lattice structures are considered as logical operations [2]. Then the non-existence of two-valued lattice homomorphisms $\varphi : \mathscr{L}(\mathscr{H}) \to 2$ implies that we can not assign values of truth or falsity for each proposition non-contextually. We show this non-existence of two-valued lattice homomorphisms by using the notion of prime filters. This idea using prime filters is due to [5], and we complete the proof.

Preliminaries

Definition 1: lattices

For a non-empty set $\mathscr{L}$, a binary relation $\leq$ on $\mathscr{L}$ is called a *partial order* if $\leq$ is reflexive, transitive and antisymmetric.

A pair $(\mathscr{L}, \leq)$ consists of a non-empty set $\mathscr{L}$ and a partial order $\leq$ on $\mathscr{L}$ is called a *partially ordered set* (or *poset* for short). A poset $(\mathscr{L}, \leq)$ is said to be *bounded* if it has both the maximum element 1 and minimum element 0 with respect to $\leq$, and denote it by $(\mathscr{L}, \leq, 0, 1)$.

A poset $(\mathscr{L}, \leq)$ is a *lattice* if there exist both the supremum (least upper bound) $a \vee b$ and the infimum (greatest lower bound) $a \wedge b$ for all $a, b \in \mathscr{L}$.

A lattice $(\mathscr{L}, \leq)$ is said to be *distributive* if the following two conditions hold:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{for all } a, b, c \in \mathscr{L};$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{for all } a, b, c \in \mathscr{L}.$$

Example 1

- For a set X, $(2^X, \subseteq, \emptyset, X)$ is a bounded distributive lattice.
- For a Hilbert space $\mathscr{H}$ with $\dim(\mathscr{H}) \geq 2$, $(\mathscr{L}(\mathscr{H}), \subseteq, \{0\}, \mathscr{H})$ is a bounded lattice but not distributive.

Definition 2: quantum logic

For a Hilbert space $\mathscr{H}$, we call the lattice of all closed linear subspaces of $\mathscr{H}$ the *quantum logic* associated with $\mathscr{H}$, and denote it by $\mathscr{L}(\mathscr{H})$.

Definition 3: lattice homomorphisms

For two bounded lattices $(\mathscr{L}_1, \leq_1, 0, 1), (\mathscr{L}_2, \leq_2, 0, 1)$, a mapping $\varphi : \mathscr{L}_1 \to \mathscr{L}_2$ is called a *lattice homomorphism* if φ satisfies the following conditions:

- $\varphi(0) = 0, \quad \varphi(1) = 1;$
- $\varphi(a \vee_1 b) = \varphi(a) \vee_2 \varphi(b), \quad \varphi(a \wedge_1 b) = \varphi(a) \wedge_2 \varphi(b)$ for all $a, b \in \mathscr{L}_1$.

In particular, if $\mathscr{L}_2 = 2 = \{0, 1\}$, we say that φ is two-valued.

A lattice homomorphism φ is called a *lattice isomorphism* if φ is bijective; and we say that $\mathscr{L}_1$ is isomorphic to $\mathscr{L}_2$.

Definition 4: filters

Let $(\mathscr{L}, \leq, 0, 1)$ be a bounded lattice. A subset $\mathcal{F}$ of $\mathscr{L}$ is called a (proper) *filter* in $(\mathscr{L}, \leq, 0, 1)$ if the following three conditions hold:

1. $0 \notin \mathcal{F}$ and $1 \in \mathcal{F}$;
2. $a \wedge b \in \mathcal{F}$ for all $a, b \in \mathcal{F}$;
3. $a \leq b$ and $a \in \mathcal{F}$ imply $b \in \mathcal{F}$ for all $a, b \in \mathscr{L}$.

A filter $\mathcal{F}$ is called a *prime filter* if for all $a, b \in \mathscr{L}$,

$$a \vee b \in \mathcal{F} \quad \text{implies} \quad a \in \mathcal{F} \quad \text{or} \quad b \in \mathcal{F}.$$

Distributive lattices have "enough" prime filters.

Fact 1: [1, Theorem 0.7]

For every bounded distributive lattice $(\mathscr{L}, \leq, 0, 1)$, there exists a bounded distributive lattice of subsets of the set of all prime filters in $(\mathscr{L}, \leq, 0, 1)$ which is isomorphic to $(\mathscr{L}, \leq, 0, 1)$.

Proof. (Sketch) Let $\mathfrak{P}(\mathscr{L})$ be the set of all prime filters in $(\mathscr{L}, \leq, 0, 1)$. A mapping

$$\varphi : a \mapsto \mathfrak{P}(a) := \{\mathcal{F} \in \mathfrak{P}(\mathscr{L}) \mid a \in \mathcal{F}\}.$$

gives a lattice isomorphism from $(\mathscr{L}, \leq, 0, 1)$ to $(\{\mathfrak{P}(a)\}_{a \in \mathscr{L}}, \subseteq, \emptyset, \mathfrak{P}(\mathscr{L}))$. ■

Lemma 1: [1, p. 9]

Let $(\mathscr{L}, \leq, 0, 1)$ be a bounded lattice and $\mathcal{F}$ a subset of $\mathscr{L}$. Then the following conditions are equivalent:

(a) $\mathcal{F}$ is a prime filter;

(b) $\mathcal{F} = \varphi^{-1}(1)$ for some two-valued lattice homomorphism $\varphi : \mathscr{L} \to 2$.

Proof. (Sketch) (a) $\implies$ (b): Suppose that $\mathcal{F}$ is a prime filter. Then a mapping $\varphi : \mathscr{L} \to 2$ defined by

$$\varphi(a) = \begin{cases} 1 & (a \in \mathcal{F}), \\ 0 & (a \in \mathscr{L} \setminus \mathcal{F}) \end{cases}$$

satisfies $\mathcal{F} = \varphi^{-1}(1)$, and one can verify that φ is a lattice homomorphism from $\mathscr{L}$ to 2.

(b) $\implies$ (a): Suppose that $\mathcal{F} = \varphi^{-1}(1)$ for some lattice homomorphism $\varphi : \mathscr{L} \to 2$. Then one can verify that $\mathcal{F}$ is a prime filter. ■

Main theorem

Theorem 1

Let $\mathscr{H}$ be a Hilbert space with $\dim(\mathscr{H}) \geq 2$. Then there exists no two-valued lattice homomorphism $\varphi : \mathscr{L}(\mathscr{H}) \to 2$.

Proof. First, we shall consider the case that $\mathscr{H}$ is a finite dimensional Hilbert space, say $\dim(\mathscr{H}) = n \, (n \in \mathbb{N}, n \geq 2)$. Suppose that there exists a two-valued lattice homomorphism $\varphi : \mathscr{L}(\mathscr{H}) \to 2$. By Lemma 1, there exists a prime filter $\mathcal{F}$ in $\mathscr{L}(\mathscr{H})$ such that $\varphi^{-1}(1) = \mathcal{F}$. We claim that there exists a $(n-1)$-dimensional linear subspace $M \in \mathscr{L}(\mathscr{H})$ such that $M \notin \mathcal{F}$. Let $\{e_i\}_{i=1}^n$ be an orthonormal system of $\mathscr{H}$. Put

$$E_i := \mathrm{span}(\{e_i\}) \quad (i = 1, \dots, n).$$

Then we have

$$\bigvee_{i=1}^n \varphi(E_i) = \varphi\left(\bigvee_{i=1}^n E_i\right) = \varphi(\mathscr{H}) = 1.$$

This implies that there exists a number $i = 1, \dots, n$ such that $\varphi(E_i) = 1$, i.e., $E_i \in \mathcal{F}$; and since $\mathcal{F}$ is a filter, we have $E_i^\perp \notin \mathcal{F}$. This shows the above claim. We fix a $(n-1)$-dimensional linear subspace $M \in \mathscr{L}(\mathscr{H})$ such that $M \notin \mathcal{F}$.

Now, consider the case that $\dim(\mathscr{H}) = 2$. Then there exists a one-dimensional linear subspace $N \in \mathscr{L}(\mathscr{H})$ such that $N \neq M$ and $N \notin \mathcal{F}$. Since $N \neq M$ and $\dim(\mathscr{H}) = 2$, we have $M \vee N = \mathscr{H} \in \mathcal{F}$. Since $\mathcal{F}$ is a prime filter, we obtain $M \in \mathcal{F}$ or $N \in \mathcal{F}$. This contradicts to the condition that $M, N \notin \mathcal{F}$.

Next, we shall consider the case that $3 \leq \dim(\mathscr{H}) < \aleph_0$. Since $\dim(M) \geq 2$ and $\dim(M^\perp) = 1$, we can take non-zero vectors $x, y, z \in \mathscr{H}$ such that

$$x, y \in M, \quad x \perp y;$$
$$z \in M^\perp.$$

Put

$$M_1 := \mathrm{span}(\{x + z\}),$$
$$M_2 := \mathrm{span}(\{y + z\}).$$

Then we have

$$M_1 \wedge M_2 = \{0\} \notin \mathcal{F},$$
$$M_1 \vee M = \mathscr{H} \in \mathcal{F},$$
$$M_2 \vee M = \mathscr{H} \in \mathcal{F}.$$

Since $\mathcal{F}$ is a prime filter, we have

$$M_1 \in \mathcal{F} \quad \text{or} \quad M \in \mathcal{F}$$

and

$$M_2 \in \mathcal{F} \quad \text{or} \quad M \in \mathcal{F}.$$

Now, since $M \notin \mathcal{F}$, we must have

$$M_1 \in \mathcal{F} \quad \text{and} \quad M_2 \in \mathcal{F}.$$

Since $\mathcal{F}$ is a filter, we obtain

$$\{0\} = M_1 \wedge M_2 \in \mathcal{F}.$$

This is a contradiction. Therefore, there exists no two-valued lattice homomorphism $\varphi : \mathscr{L}(\mathscr{H}) \to 2$ for the case that $2 \leq \dim(\mathscr{H}) < \aleph_0$.

Finally, we consider the case that $\mathscr{H}$ is a infinite dimensional Hilbert space. Then the quantum logic $\mathscr{L}(\mathscr{H}_n)$ associated with a n-dimensional Hilbert space $\mathscr{H}_n$ $(n \in \mathbb{N}, \ n \geq 2)$ can be embedded into $\mathscr{L}(\mathscr{H})$ by an embedding $\iota : \mathscr{L}(\mathscr{H}_n) \to \mathscr{L}(\mathscr{H})$. If there exists a two-valued lattice homomorphism $\varphi : \mathscr{L}(\mathscr{H}) \to 2$, then the composite $\varphi \circ \iota : \mathscr{L}(\mathscr{H}_n) \to 2$ gives a two-valued lattice homomorphism, which is impossible as we have shown in the above. Consequently, for any Hilbert space $\mathscr{H}$ with $\dim(\mathscr{H}) \geq 2$, there exists no two-valued lattice homomorphism $\varphi : \mathscr{L}(\mathscr{H}) \to 2$. ■

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