On Noncontextual, Non-Kolmogorovian Hidden Variable Theories

Samuel C. Fletcher[1]
Benjamin H. Feintzeig[2]

[1]Department of Philosophy
University of Minnesota, Twin Cities
&
Munich Center for Mathematical Philosophy
Ludwig-Maximilians-Universität

[2]Department of Philosophy
University of Washington

5 June, 2017

Motivation

- Fine (1982), Pitowsky (1989): satisfaction of Bell inequalities equivalent to existence of Kolmogorovian hidden variable model.
  - Hence motivation for using slight generalizations of classical probability.
- But what about the Kochen-Specker theorem? The conjunction of
  - value realism,
  - value definiteness,
  - and noncontextuality

  is incompatible with QM.
- The KS theorem bears on many non-Kolmogorovian hidden variable models that seem to meet these conditions!

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Outline

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Standard Non-Relativistic QM

Definition

A **simple quantum mechanical experiment** is a triple $(\mathcal{H}, \psi, \mathcal{O}_n)$, where $\mathcal{H}$ is a Hilbert space, $\psi \in \mathcal{H}$ is a unit vector, and $\mathcal{O}_n = \{P_1, \ldots, P_n\}$ is a collection of n projection operators on $\mathcal{H}$.

Kolmogorovian Probability

Definition

A **(σ-)algebra** Σ for a set X is a nonempty subset of $\wp(X)$ such that:

1. for all $A \in \Sigma$, $X - A \in \Sigma$; and
2. for all $A, B \in \Sigma$, $A \cup B \in \Sigma$.

Definition

A **classical probability space** is a triple (X, Σ, μ), where Σ is a (σ-)algebra for X and $\mu : \Sigma \to [0, 1]$ is such that

1. $\mu(X) = 1$; and
2. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

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Bell's Theorem is about Classical Probability

Definition

A **(restricted) classical probability space representation** for a simple quantum mechanical experiment $(\mathcal{H}, \psi, \mathcal{O}_n)$ is a classical probability space (X, Σ, μ) and a map $E : \mathcal{O}_n \to \Sigma$ satisfying both of the following conditions:

1. for each $P_i \in \mathcal{O}_n$, $\mu(E(P_i)) = \langle\psi|P_i|\psi\rangle$; and
2. for each $P_i, P_j \in \mathcal{O}_n$, if $[P_i, P_j] = 0$, then $\mu(E(P_i) \cap E(P_j)) = \langle\psi|P_iP_j|\psi\rangle$.

Theorem (Fine, 1982; Pitowsky, 1989)

The probabilities for outcomes of a simple quantum mechanical experiment satisfy all Bell-type inequalities iff the experiment has a classical probability space representation.

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Corollary

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The Kochen-Specker Theorem

Theorem (Kochen-Specker)

For any Hilbert space $\mathcal{H}$ of dimension at least 3, there is a finite collection of projection operators $\mathcal{O}_n$ on it such that there is no function $f : \mathcal{O}_n \to \{0, 1\}$ that assigns 1 to exactly one element of every subset of $\mathcal{O}_n$ whose elements are mutually orthogonal and span $\mathcal{H}$.

Definition

A **KS-witness** is a simple quantum mechanical experiment $(\mathcal{H}, \psi, \mathcal{O}_n)$ such that $\dim(\mathcal{H}) \geq 3$ and there is no function $f : \mathcal{O}_n \to \{0, 1\}$ that assigns 1 to exactly one element of every subset of $\mathcal{O}_n$ whose elements are mutually orthogonal and span $\mathcal{H}$.

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Weak Representations

Definition

A **weak (noncontextual) probability space** is an ordered triple (X, Σ, μ), where X is a nonempty set, $\Sigma \subseteq \wp(X)$ is nonempty, and $\mu : \Sigma \to Y \supseteq [0, 1]$.

Definition

A **weak hidden variable representation** for a simple quantum mechanical experiment $(\mathcal{H}, \psi, \mathcal{O}_n)$ is a weak probability space (X, Σ, μ) and a map $E : \mathcal{O}_n \to \Sigma$ satisfying both of the following conditions:

1. for each $P_i \in \mathcal{O}_n$, $\mu(E(P_i)) = \langle \psi | P_i | \psi \rangle$; and
2. for each $P_i, P_j \in \mathcal{O}_n$, if $P_i \perp P_j$, then $E(P_i) \cap E(P_j) \in \Sigma$ and $\mu(E(P_i) \cap E(P_j)) = \langle \psi | P_i P_j | \psi \rangle = 0$.

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The Central Theorem

Theorem

No KS-witness $(\mathcal{H}, \psi, \mathcal{O}_n)$ has a weak hidden variable representation (X, Σ, μ, E) satisfying both of the following:

Weak Classicality (WC) *if $Q \subseteq \mathcal{O}_n$ contains only mutually orthogonal operators spanning $\mathcal{H}$, then $(X, \Sigma_Q, \mu_{|\Sigma_Q})$ is a classical probability space, where $\Sigma_Q \subseteq \Sigma$ is the smallest (σ-)algebra for X containing $\{E(P_i) : P_i \in Q\}$; and*

No Finite Null Cover ($\neg$FNC) *there is no collection $B_1, \ldots, B_m \in \Sigma$ such that $\mu(B_i) = 0$ for all $i \in \{1, \ldots, m\}$ and $\bigcup_{i=1}^{m} B_i = X$.*

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The Central Corollary

Corollary

No weak hidden variable theory—a noncontextual assignment of a weak hidden variable representation to each simple quantum mechanical experiment—can satisfy both WC and $\neg$*FNC.*

Outline

Application: Generalized Probability Spaces (Suppes/Fine/Gudder)

Definition

A **(σ-)additive class** Σ for a set X is a nonempty subset of $\wp(X)$ such that:

1. for all $A \in \Sigma$, $X - A \in \Sigma$; and
2. for all disjoint $A, B \in \Sigma$, $A \cup B \in \Sigma$.

Definition

A **generalized probability space** is a triple (X, Σ, μ), where Σ is a (σ-)additive class for X and $\mu : \Sigma \to [0, 1]$ is such that

1. $\mu(X) = 1$; and
2. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Application: Negative Probability Spaces (Hartle/Kronz/et al.)

Definition

A **(σ-)algebra** Σ for a set X is a nonempty subset of $\wp(X)$ such that:

1. for all $A \in \Sigma$, $X - A \in \Sigma$; and
2. for all $A, B \in \Sigma$, $A \cup B \in \Sigma$.

Definition

A **negative probability space** is a triple (X, Σ, μ), where Σ is a (σ-)algebra for X and $\mu : \Sigma \to \mathbb{R}$ is such that

1. $\mu(X) = 1$; and
2. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Application: Complex Probability Spaces (Youssef/Srinivasan/et al.)

Definition

A **(σ-)algebra** Σ for a set X is a nonempty subset of $\wp(X)$ such that:

1. for all $A \in \Sigma$, $X - A \in \Sigma$; and
2. for all $A, B \in \Sigma$, $A \cup B \in \Sigma$.

Definition

A **complex probability space** is a triple (X, Σ, μ), where Σ is a (σ-)algebra for X and $\mu : \Sigma \to \mathbb{C}$ is such that

1. $\mu(X) = 1$; and
2. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Application: Quantum Measure Theory (Sorkin/Gudder)

Definition

A **(σ-)algebra** Σ for a set X is a nonempty subset of $\wp(X)$ such that:

1. for all $A \in \Sigma$, $X - A \in \Sigma$; and
2. if $A, B \in \Sigma$, then $A \cup B \in \Sigma$.

Definition

A **quantum measure space** is a triple (X, Σ, μ), where Σ is a (σ-)algebra for X and $\mu : \Sigma \to [0, 1]$ is such that

1. $\mu(X) = 1$; and
2. for any mutually disjoint $A, B, C \in \Sigma$, $\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C)$.

Application: Upper (Lower) Probability Spaces (Suppes et al.)

Definition

A **(σ-)algebra** Σ for a set X is a nonempty subset of $\wp(X)$ such that:

1. for all $A \in \Sigma$, $X - A \in \Sigma$; and
2. for all $A, B \in \Sigma$, $A \cup B \in \Sigma$.

Definition

An **upper (lower) probability space** is a triple (X, Σ, μ), where Σ is a (σ-)algebra for X and $\mu : \Sigma \to [0, 1]$ is such that

1. $\mu(X) = 1$; and
2. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) \leq (\geq)\ \mu(A) + \mu(B)$.

Application of Central Theorem

Theorem

Generalized probability spaces, negative probability spaces, complex probability spaces, quantum measure spaces, and upper (lower) probability spaces are all weak probability spaces.

Corollary

No KS-witness has a weak hidden variable representation with a generalized probability space, negative probability space, complex probability space, quantum measure space, or upper (lower) probability space that satisfies both WC and $\neg$FNC.

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Generalized probability spaces, negative probability spaces, complex probability spaces, quantum measure spaces, and upper (lower) probability spaces are all weak probability spaces.

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Pyrrhic Victory for Classical Logic?

- Nonclassical probability sometimes motivated by a desire to maintain the "classical logic" of noncontextuality.
- But a FNC would then require a finite disjunction of "negative sureties/falsities" to be a "surety/truth".
- How could the "probabilities" of such a model provide guidance for belief? Credences set by them can be Dutch booked.

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Against Finite Null Covers

Definition

A **Dutch book** in a weak probability space (X, Σ, μ) is a collection of events $B_1, \ldots, B_m \in \Sigma$ along with numbers $s_1, \ldots, s_m \in \mathbb{R}$ such that for any $x \in X$,

$$\sum_{i=1}^{m} s_i(\chi_{B_i}(x) - \mu[B_i]) < 0$$

Theorem

If a weak probability space (X, Σ, μ) contains a finite null cover $\{B_1, \ldots, B_m\}$, then it contains a Dutch book $\{B_1, \ldots, B_m\}$ with any $s_i < 0$ for all i.

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Settleable Bets

Hartle (2004) and Gell-Mann and Hartle (2012) consider a related problem for negative probabilities:

- Only *settleable* bets, whose outcomes an agent is guaranteed to know, are legitimate.

However, each B_i in the null cover, and the whole collection, is settleable.

- This is despite the disjunction of some elements of the cover not being settleable.

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Future Work

- Only need "enough" value-definiteness to get a KS-witness; thus might apply to some models (e.g., modal interpretations) that reject *general* value-definiteness.
- Other nonclassical probability models?
- Any viable interpretation of a FNC?
- Hierarchy of no-go theorems?

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For more information, see the full paper:

"On Noncontextual, Non-Kolmogorovian Hidden Variable Theories." *Foundations of Physics* 47.2 (2017): 294–315.

DOI:10.1007/s10701-017-0061-z

arXiv:1608.03518

Both SCF and BHF acknowledge the support of National Science Foundation Graduate Research Fellowships.