Almost Equivalent Paradigms of Contextuality

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1. Introduction

Contextuality of quantum mechanics entails the impossibility of assigning predetermined outcomes to observables, independently from the method of observation. We demonstrate how the sheaf-theoretic approach to contextuality by Abramsky and Brandenburger [1] translates to the equivalence-based approach by Spekkens [3]. This translation defines a categorical isomorphism. The two approaches describe non-contextuality for general operational theories, independently from the quantum formalism.

2. The Sheaf-Theoretic Approach

Contextuality as a global inconsistency of joint measurements

Probability distributions over joint measurements form a presheaf

$$\mathcal{D}_{\mathbb{P}}\mathcal{E} : \mathcal{P}(X)^{op} \to Set$$

Here, X is a set of measurements, O^X the set of functions from X to an outcome set O, $\mathcal{M}$ a cover of maximal joint measurements of X, and $\mathcal{D}_{\mathbb{P}}(X)$ is the set of probability distributions over X. We obtain the presheaf by composing the sheaf $\mathcal{E} : U \mapsto O^U$ with the functor

$$\mathcal{D}_{\mathbb{P}}(X) \xrightarrow{\mathcal{D}_R(f)} \mathcal{D}_{\mathbb{P}}(Y) :: d \mapsto [y \mapsto \sum_{f(x)=y} d(x)$$

A **state** σ is given by a distribution $\sigma_C \in \mathcal{D}_{\mathbb{P}}\mathcal{E}(C)$, for each measurement context $C \in \mathcal{M}$. It is **non-contextual** if it has a global section $d \in \mathcal{D}_{\mathbb{P}}\mathcal{E}(X)$. This is a joint distribution such that $\forall C \in \mathcal{M}$

$$\sum_{s' \in \mathcal{E}(X)s|_C = s} d(s') = \sigma_C(s)$$

An **empirical model** $(X, \mathcal{M}, S)$, where S is the set of states for $X, \mathcal{M}$, is **non-contextual** if all states are non-contextual.

3. The Equivalence-Based Approach

Contextuality as an ontological distinction between operationally equivalent operations

Operational theories (P, M, d) are sets P, M, of preparations and measurements, with probability distributions

$$d(p, m) : O \to [0, 1] \qquad \forall (p, m) \in P \times M$$

Ontological representations (Ω, μ, ξ), consist of a set of ontological values Ω, together with a set of probability distributions

$$\mu = \{\mu_p : \Omega \to [0,1]\}_{p \in P}$$

$$\xi = \{\xi_m(\lambda) : O^m \to [0,1]\}_{\lambda \in \Omega, m \in M}$$

$$\text{s.t. } \int_\Omega \xi_m(\lambda)\mu_p(d\lambda) = d_A(p, m) \quad \forall p \in P, m \in M$$

An ontological representation is **non-contextual** if it is defined on statistical equivalence classes of preparations and measurements: $\forall n \in M, q \in P$

$$p \sim p' \quad \Leftrightarrow \quad d(p, n) = d(p', n)$$

$$m \sim m' \quad \Leftrightarrow \quad d(q, m) = d(q, m')$$

$$(m, k) \sim (m', k') \quad \Leftrightarrow \quad d(q, m)(k) = d(q, m')(k')$$

Preparation non-contextuality:
$p \sim p' \Rightarrow \mu_p = \mu_{p'}$

Measurement non-contextuality:
$(m, k) \sim (m', k') \Rightarrow \xi_{k,m} = \xi_{k',m'}$.

4. Canonical Ontological Representations

Statistical equivalence does not affect contextuality in the sheaf approach.

A set of global sections $\{d^\sigma\}_{\sigma \in S}$ defines a non-contextual ontological representation $R^{NC}(A)$, based on statistical equivalence classes $[m] \in X/\sim$ where $\tilde{d}^{\tilde{\sigma}}([s]) := d^\sigma(s)$:

$$\Omega_A^{NC} := \mathcal{E}(X/\sim), \qquad \mu_\sigma^{NC}(s) := \tilde{d}^{\sigma}([s]), \qquad \xi_m^{NC}(s)(k) := \delta_{[s]|_{[m]}([m]),[k]}$$

We call this the 'canonical' ontological representation.

5. Joint Measurements

A set of N measurements $\{m_1, m_2, ..., m_N\}$ is **jointly measurable** if there exists a measurement m with the following features [2]:

- The outcome set of m is the Cartesian product of the outcome sets of $\{m_1, ..., m_N\}$
- The outcome distributions are recovered as marginals of the outcome distribution of m.

$$\forall S, \forall p : p(k_S | m_S; p) = \sum_{k \in O^m : \pi_S(k) = k_S} p(k|m; p)$$

Here, π_S is the projection function on the subspace $\mathcal{M}(m_S) \subset \mathcal{E}(m)$

6. Induced Global Sections

Tracing out different contexts gives statistically equivalent measurements. One can induce that non-contextuality implies **parameter independence**:

$$\text{For two joint measurements } m, n, \qquad \xi_m|_{m \cap n} = \xi_n|_{m \cap n}$$

Here, $\xi_m|_{m'}(k')(\lambda) := \sum_{k : p_{m'}(k) = k'} \xi_m(k)(\lambda)$, for the projection $p_{m'} : O^m \to O^{m'}$.

An ontological representation is **factorizable** when for joint measurements $m = (m_1, ..., m_n)$ we can write $\xi_m(\lambda)(o) \prod_{i=1,...,n} \xi_{m_i}(\lambda)(\pi_i(o))$.

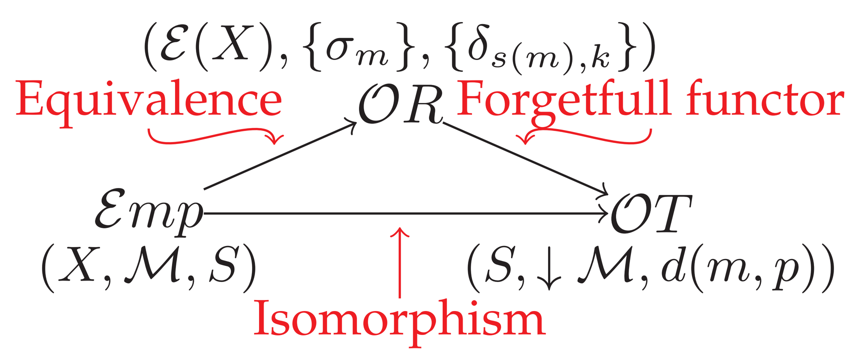
For every factorizable, non-contextual ontological representation B, there exists an empirical model A with states $\{\sigma_p\}$ and global sections d

$$\sigma_p(s) = \int_\Omega \xi_m(\lambda)(s(m))\mu_P(d\lambda), \qquad d(s) = \int_\Omega \prod_{m \in Min(M_B)} \xi_m(\lambda)\mu_P(d\lambda)$$

One can show that $R^{NC}(A)$ and B realise the same operational theory.

7. Theorem

A categorical isomorphism



For models with **sharp measurements** we have **functors** between the categories $\mathcal{E}mp$, of empirical models; $\mathcal{O}R$, of ontological representations; and $\mathcal{O}T$, of operational theories, which **preserve non-contextuality**.

8. Conclusions

- Sheaf-theoretic non-contextuality corresponds to the existence of a factorizable non-contextual ontological representation in the equivalence-based model.
- The two formalisms agree on classic examples of contextuality, as well as contextuality for preparations and for unsharp measurements introduced in [3].
- The canonical ontological representation provides a framework for contextuality in noise-free quantum circuits.

References

[1] S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics 13*, 2011.

[2] Y. Liang, R. W. Spekkens, and H. M. Wiseman. Specker's parable of the overprotective seer: A road to contextuality, nonlocality and complementarity. *Physics Reports*, 2011. vol 506, issues 1-2, pp 1-39.

[3] R. W. Spekkens. Contextuality for preparations, transformations, and unsharp measurements. *A Physical Review*, 71, 2005.