1. **Introduction**

Contextuality of quantum mechanics entails the impossibility of assigning predetermined outcomes to observables, independently from the method of observation. We demonstrate how the sheaf-theoretic approach to contextuality by Abramsky and Brandenburger translates to the equivalence-based approach by Spekkens. This translation defines a categorical isomorphism. The two approaches describe non-contextuality for general operational theories, independently from the quantum formalism.

2. **The Sheaf-Theoretic Approach**

Contextuality as a global inconsistency of joint measurements

Probability distributions over joint measurements form a presheaf

\[ \mathcal{D}_X: P(X)^{op} \to \text{Set} \]

Here, \( X \) is a set of measurements, \( O^X \) the set of functions from \( X \) to an outcome set \( O \), and \( C/M \) a cover of maximal joint measurements of \( X \), and \( \mathcal{D}_X(Y) \) is the set of probability distributions over \( Y \). We obtain the presheaf by composing with a sheaf \( \mathcal{E}: U \to \text{Set} \) with the functor \( \mathcal{D}_Y: P(Y)^{op} \to \text{Set} \), where \( S \) is the set of states for \( X \), \( M \), is non-contextual if all states are non-contextual.

3. **The Equivalence-Based Approach**

Contextuality as an ontological distinction between operationally equivalent representations

A set of measurements \( \{ m_1, m_2, \ldots, m_N \} \) is jointly measurable if there exists a measurement \( m \) with the following features [2]:

- The outcome set of \( m \) is the Cartesian product of the outcome sets of \( \{ m_1, \ldots, m_N \} \).
- The outcome distributions are recovered as marginals of the outcome distribution of \( m \).

\[ \forall S, \forall p: p(k|s|m;p) = \sum_{k \in O^m} \pi_S(k|m;p) \]

Here, \( \pi_S \) is the projection function on the subspace \( M(m_S) \subset \mathcal{E}(m) \).

4. **Canonical Ontological Representations**

Statistical equivalence does not affect contextuality in the sheaf approach.

A set of global sections \( \{ d^p \}_{p \in S} \) defines a non-contextual ontological representation \( R_{NC}(A) \), based on statistical equivalence classes \( [m] \in X/ \sim \) where \( d^p([s]) := d^p(s) \):

\[ \Omega_{da} := \mathcal{E}(X/ \sim), \quad \mu_{sa}^{NC}(s) := d^s([s]), \quad \xi_{ma}^{NC}(s)(k) := \delta_{d^s|[m]([m]),[k]} \]

We call this the ‘canonical’ ontological representation.

5. **Joint Measurements**

A set of \( N \) measurements \( \{ m_1, m_2, \ldots, m_N \} \) is jointly measurable if there exists a measurement \( m \) with the following features [2]:

- The outcome set of \( m \) is the Cartesian product of the outcome sets of \( \{ m_1, \ldots, m_N \} \).
- The outcome distributions are recovered as marginals of the outcome distribution of \( m \).

\[ \forall S, \forall p: p(k|s|m;p) = \sum_{k \in O^m} \pi_S(k|m;p) \]

Here, \( \pi_S \) is the projection function on the subspace \( M(m_S) \subset \mathcal{E}(m) \).

6. **Induced Global Sections**

Tracing out different contexts gives statistically equivalent measurements. One can induce that non-contextuality implies parameter independence:

For two joint measurements \( m, n \), \( \xi_m|n(m) = \xi_n|m(m) \).

Here, \( \xi_m|n(m) := \sum_{k,p_{m,n}(k)=k} \xi_m(k)(\lambda), \) for the projection \( p_{m,n} : O \to O^{m,n} \).

An ontological representation is **factorizable** when for joint measurements \( m = (m_1, \ldots, m_m) \) we can write \( \xi_m(\lambda(o)) = \lambda_{t}(\lambda(o)) \).

For every factorizable, non-contextual ontological representation \( B \), there exists an empirical model \( A \) with states \( \{\sigma_p\} \) and global sections \( \sigma_p(s) := \int_{\Omega^A} \xi_m(\lambda(s(m)))|\mu_p(d\lambda), \quad d(s) := \int_{\Omega^A} \prod_{m \in M(m_M)} \xi_m(\lambda)||\mu_p(d\lambda) \)

One can show that \( R_{NC}(A) \) and \( B \) realise the same operational theory.

7. **Theorem**

A categorical isomorphism

\[ (\mathcal{E}(X), \{\sigma_m\}, \{\xi_m(a,m)\}) \quad \text{Equivalence} \quad \text{OR Forgetful functor} \]

\[ (S, \downarrow M, d(m,p)) \quad \text{OT} \quad \text{Isomorphism} \]

For models with sharp measurements we have functors between the categories \( \mathcal{E} \) of empirical models; \( \mathcal{O} \), of ontological representations; and \( \text{OT} \), of operational theories, which preserve non-contextuality.

8. **Conclusions**

- Sheaf-theoretic non-contextuality corresponds to the existence of a factorizable non-contextual ontological representation in the equivalence-based model.
- The two formalisms agree on classic examples of contextuality, as well as for transformations and for unsharp measurements introduced in [3].
- The canonical ontological representation provides a framework for contextuality in noise-free quantum circuits.

References